Homework 8 Solutions

1. Let \( a, b \in F \) be nonzero elements such that \( a + b \neq 0 \). Prove that the quadratic forms \( \langle a, b \rangle \) and \( \langle a + b, ab(a + b) \rangle \) are isomorphic.

**Proof.** Consider the map \( T : F^2 \to F^2 \) by \( T(x_1, x_2) = (x_1 - bx_2, x_1 + ax_2) \). We see if \( Q_1 = \langle a, b \rangle \) and \( Q_2 = \langle a + b, ab(a + b) \rangle \), then

\[
Q_1(T(x_1, x_2)) = Q_1(x_1 - bx_2, x_1 + ax_2) = a(x_1 - bx_2)^2 + b(x_1 + ax_2)^2
\]

Next we could take column one scaled by \( -b \) and add it to column two, placing it in column two, which has matrix \( \begin{bmatrix} 1 & -b \\ a & 1 \end{bmatrix} \) (on the left). We compute

Next we could take column one scaled by \( \frac{-b}{a+b} \) and add it to column two, placing it in column two, which has matrix \( \begin{bmatrix} 1 & 0 \\ -b \frac{1}{a+b} & 1 \end{bmatrix} \) (on the right). We compute

Lastly we multiply the second column by \( a + b \), with the corresponding row operation gives us

\[
\begin{bmatrix} 1 & 0 \\ 0 & a+b \end{bmatrix} \begin{bmatrix} 0 \\ \frac{ab}{a+b} \end{bmatrix} = \begin{bmatrix} a+b \\ 0 \end{bmatrix}
\]

which is the result. Note that \( T \) comes from

\[
\begin{bmatrix} 1 & 0 \\ 0 & a+b \end{bmatrix} \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} a+b \\ ab(a+b) \end{bmatrix}
\]

2. Determine all isometries of the quadratic form \( Q(x, y) = xy \) on \( F^2 \).

**Solution.** Suppose \( T : F^2 \to F^2 \) is an isometry for \( Q \), so \( Q(T(x, y)) = xy \). If \( T(x, y) = (ax + by, cx + dy) \), this is

\[
(ax + by)(cx + dy) = acx^2 + (ad + bc)xy + bdy^2 = xy
\]

which gives the equations \( ac = 0, ad + bc = 1, bd = 0 \), and \( ad - bc \neq 0 \) (as \( T \) must be an isomorphism). Suppose \( a = 0 \). Then, \( bc = 1 \), so \( c = b^{-1} \) and \( b \neq 0 \). Thus, \( d = 0 \), so \( T(x, y) = (by, b^{-1}x) \). If \( c = 0 \), then
ad = 1 so d = a^{-1} and a \neq 0. Thus, b = 0 as well and T(x, y) = (ax, a^{-1}y). It is clear that such T preserve Q, and so these are the desired isometries.

3. Let B be a (symmetric) bilinear form on a vector space V and let W be a subspace of V. Prove that W^\perp = ((W^\perp)^\perp)^\perp.

Proof. We first show for any subset X, X \subseteq (X^\perp)^\perp. Indeed, fix x \in X and suppose y \in X^\perp. Then, B(x, y) = B(y, x) = 0, so as y was arbitrary in X^\perp we have x \in (X^\perp)^\perp. Applying this to X = W^\perp we have W^\perp \subseteq ((W^\perp)^\perp)^\perp. Now, we note for any subsets A, C \subseteq V with A \subseteq C, we have A^\perp \supseteq C^\perp. Indeed, if y \in C^\perp, then B(y, c) = B(c, y) = 0 for all c \in C. As A \subseteq C, we see B(y, a) = B(a, y) = 0 for all a \in A, so y \in A^\perp. Thus, applying this to W \subseteq (W^\perp)^\perp (from the first claim with X = W) we have W^\perp \supseteq ((W^\perp)^\perp)^\perp.

4. Prove that B(X, Y) = tr(XY) is a bilinear form on M_{2 \times 2}(F). Investigate if B is degenerate or not.

Proof. We note

\begin{align*}
B(aX_1 + X_2, bY_1 + Y_2) &= tr((aX_1 + X_2)(bY_1 + Y_2)) = tr(abX_1Y_1 + aX_1Y_2 + bX_2Y_1 + X_2Y_2) \\
&= abtr(X_1Y_1) + atr(X_1Y_2) + btr(X_2Y_1) + tr(X_2Y_2) \\
&= abB(X_1, Y_1) + abB(X_1, Y_2) + bB(X_2, Y_1) + B(X_2, Y_2)
\end{align*}

so B is bilinear. Let e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, and e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} and \sigma = \{e_1, e_2, e_3, e_4\}. Then,

\begin{bmatrix}
B_{ij}
\end{bmatrix}_\sigma = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}

so, as this is nonsingular, B is nondegenerate.

5. Let S be a subset of an inner product vector space V and W = span(S). Prove v \in W^\perp if an only if v \perp w for all w \in S.

Proof. Suppose v \in W^\perp. Then, as S \subseteq W, S^\perp \supseteq W^\perp, so v \in S^\perp. That is, v \perp w for all w \in S. Now, suppose v \perp w for all w \in S. Let x \in W. Then, x = \sum_{k=1}^m a_k w_k for some w_1, ..., w_m \in S. Hence,

\langle v, x \rangle = \langle v, \sum_{k=1}^m a_k w_k \rangle = \sum_{k=1}^m \overline{a_k} \langle v, w_k \rangle = 0

as w_k \in S for each k. As x was arbitrary, we conclude v \in W^\perp.

6. Let T be a linear operator on an inner product space such that \|T(v)\| = \|v\| for all v \in V. Prove that T is one-to-one.

Proof. Suppose v \in N(T). Then, T(v) = 0, so \|T(v)\| = 0. Hence, \|v\| = 0 by hypothesis, so v = 0 as this is an inner product space.

7. Let T be a linear operator on an inner product space such that \|T(v)\| = \|v\| for all v \in V. Prove that \langle T(v), T(u) \rangle = \langle v, u \rangle for all v, u \in V.

Proof. We use the polarization identity \langle x, y \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2 (or, if F = \mathbb{R}, \langle x, y \rangle = \frac{1}{4} \|x + y\|^2 - \|x - y\|^2) which follows by expanding the norms and sum on the right. Thus,

\begin{align*}
\langle T(v), T(u) \rangle &= \frac{1}{4} \sum_{k=0}^3 i^k \|T(v) + i^k T(u)\|^2 \\
&= \frac{1}{4} \sum_{k=0}^3 i^k \|T(v + i^k u)\|^2 \\
&= \frac{1}{4} \sum_{k=0}^3 i^k \|v + i^k u\|^2 = \langle v, u \rangle
\end{align*}
or, in the real case,

\[
\langle T(v), T(u) \rangle = \frac{1}{4} (\|T(v) + T(u)\|^2 + \|T(v) - T(u)\|^2) = \frac{1}{4} (\|T(v + u)\|^2 + \|T(v - u)\|^2) = \frac{1}{4} (\|v + u\|^2 + \|v - u\|^2) = \langle v, u \rangle
\]