131BH- Solutions

Homework 5

Problem 9 Assume $f$ is uniformly continuous. Given $\epsilon > 0$ there is $\delta > 0$ such that $d_X(x,y) < \delta \implies d_Y(f(x),f(y)) < \epsilon/2$. Assume $\text{diam} E \leq \delta$. Then by the above $\sup \{d_Y(f(x), f(y)) : x, y \in E\} \leq \epsilon/2 < \epsilon$. Conversely, if the diameter condition holds for $\epsilon$ and $\delta$, then when $d_X(x,y) < \delta$, $E = \{x, y\}$ has $\text{diam} E < \delta$, so that $d_Y(f(E)) = d_Y(f(x), f(y)) < \epsilon$ and $f$ satisfies the definition of uniform continuity.

Problem 10 Assume $f$ is continuous from the compact space $X$ to the metric space $Y$, but assume $f$ is not uniformly continuous. Then there exists $\epsilon > 0$ for which the definition of UC fails. That means there are $x_n \in X$ and $y_n \in Y$ such that $d_X(x_n, y_n) < 1/n$ but $d_Y(f(x_n), f(y_n)) \geq \epsilon$. By compactness $\{x_n\}$ has a convergent subsequence $x_{n_j} \to x \in X$ and since $d_X(x_{n_j}, y_{n_j}) \to 0$, we also have $y_{n_j} \to x$. But then $\epsilon \leq \lim sup d_Y(f(x_{n_j}), f(y_{n_j})) = d_Y(f(x), f(x)) = 0$, a contradiction!

Problem 11 Let $f : X \to Y$ be uniformly continuous and let $\{x_n\}$ be a Cauchy sequence in $X$. For any $\epsilon > 0$ there is $\delta > 0$ such that $d_X(x_n, x_m) < \delta \implies d_Y(f(x_n), f(y_m)) < \epsilon$ by the definition of UC and for $\delta$ there is $N > 0$ such that $n > N$ and $m > N \implies d_X(x_n, x_m) < \delta$ by the definition of Cauchy sequence. Thus $d_Y(f(x_n), f(x_m)) < \epsilon$ whenever $n > N$ and $m > N$.

See Problem 13 for an application of Problem 11.

Note that the converse of Problem 11 is also true.

Problem 12 Easy from the definition. Let $X, Y$ and $Z$ be metric spaces with metrics $d_X, d_Y$ and $d_Z$. Assume $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous. Then given $\epsilon > 0$ there is $\delta > 0$ such that $d_Y(g(y_1), g(y_2)) < \delta \implies d_Z(g(y_1), g(y_2)) < \epsilon$ and there is $\eta > 0$ such that $d_X(x_1, x_2) < \eta \implies d_Y(f(x_1), f(x_2)) < \eta$. Combining these for $y_j = f(x_j)$ we get $d_X(x_1, x_2) < \eta \implies d_Z(g(f(x_1)), g(f(x_2))) < \epsilon$.

Problem 13 For each $p \in X$ let $V_n(p) = E \cap \{q : d_X(g,q) < 1/n\}$. By Exercise 9 and Theorem 3.10 in Rudin $\text{diam} f(V_n(p)) \to 0$ ($n \to \infty$). But each $f(V_n(p))$ is compact because it is closed and bounded. Hence Theorem 3.10 in Rudin also shows that $\bigcap f(V_n(p))$ consists of exactly one point, which we call $g(p)$.

That defines a function $g : X \to \mathbb{R}$ such that $g(x) = f(x)$ when $x \in E$. We must show that $g$ is continuous. But given $\epsilon > 0$ there is $\delta > 0$ such
that if \( x, y \in E \) and \( d_X(x, y) < \delta, |f(x) - f(y)| < \varepsilon/3 \). Let \( d_X(p, q) < \delta/3 \). Take 1/n < \delta/3 and take \( x \in V_n(p) \). Then \( \text{diam}(V_n(p)) \leq \varepsilon/3 \), so that \( |g(p) - f(x)| \leq \varepsilon/3 \). Similarly, if \( y \in V_n(q) \), \( |f(y) - g(q)| \leq \varepsilon/3 \). But also \( d_X(x, y) < \delta \). Hence

\[
|f(x) - f(y)| < \varepsilon
\]

Note that this also shows \( g \) is uniformly continuous.

The above argument also works if \( f \) has range any compact metric space, because Theorem 3.10 is valid for compact metric spaces.

The theorem is also true if \( f \) has range a complete metric space, and the proof based on Problem 11 works in that case. To use Problem 11, take \( x_n \in V_n(p) \). Then \( \{x_n\} \) is a Cauchy sequence, and by Problem 11, \( f(x_n) \) is a Cauchy sequence. Let \( g(p) = \lim f(x_n) \) (range is complete). Then the \( \varepsilon/3 \) proof that \( g \) is continuous works in this case as well.

If \( X = [0, 1], E = Y = (0, 1] \) and \( f(x) = x \) then \( E \) is dense in \( X \) and \( f : E \rightarrow Y \) is uniformly continuous but \( f \) has no continuous extension to \( Y \).

Note that this exercise, plus the fact that a continuous function on a compact set is uniformly continuous, gives another proof of Exercise 5 at least for bounded closed sets.

**Problem 14.** Note that this is easy if \( f(0) = 0 \) or if \( f(1) = 1 \). Thus we may assume \( f(0) > 0 \) and \( f(1) < 1 \). Draw a picture. Now let

\[
F(x) = f(x) - x.
\]

Then \( F \) is continuous on \( I \), \( F(0) > 0 \) and \( F(1) < 1 \). By the intermediate value theorem 4.23, \( F(x) = 0 \) for some \( x \in I \). But then \( f(x) = x \).

Note: This is called the Brower Fixed Point Theorem. It is also true for maps from the unit cube of \( \mathbb{R}^n \) into itself, but the proof is harder. One proof uses algebraic topology, a second proof uses advanced calculus cleverly.

**Problem 16** Both \( [x] \) and \( \{x\} = x - [x] \) are continuous except at the integers, where both have discontinuities of the first kind.

**Problem 17** The hint says it all.