Name: SOLUTIONS

Write your answers on your exam. You may remove the scratch paper at the end of your exam. All questions have equal value.

1
2
3
4
Total
1. Let \( \{x_n\} \) be a bounded sequence of real numbers, let \( E \) be the set of all limits of subsequences of \( \{x_n\} \), and let \( A = \sup E \).

(a) Prove that \( A \in E \), i.e. prove that \( A \) is the limit of a subsequence of \( \{x_n\} \).

For each \( j = 1, 2, \ldots \), there is a limit \( y_j \) of a subsequence of \( \{x_n\} \) such that \( A - \frac{1}{j} < y_j < A \), and there is \( n_j \) such that \( |x_{n_j} - y_j| < \frac{1}{2^j} \) and (if \( j \geq 2 \)) \( n_j > n_{j-1} \). Then \( \{x_{n_j}\} \) is a subsequence such that

\[
\lim x_{n_j} = \lim y_j = A.
\]

(b) Prove that

\[
A = \inf \{B : \text{ for all } \epsilon > 0, x_n > B + \epsilon \text{ for only finitely many } n \}.
\]

Write

\[
S = \{B : \text{ for all } \epsilon > 0, x_n > B + \epsilon \text{ for only finitely many } n \}.
\]

If \( B \in S \) and \( C > B \) then \( C \notin E \), because \( x_n > \frac{B + C}{2} \) for only finitely many \( n \). Hence \( B \) is an upper bound for \( E \), and \( B \geq A \). However, for any \( \epsilon > 0 \), \( A + \epsilon \notin E \), so that \( A + \epsilon \in S \). Hence \( \inf S \leq A + \epsilon \).
2. Let \( a_n \geq a_{n+1} \geq 0 \) be a decreasing sequence of positive real numbers and define

\[ b_n = n a_n^2. \text{ (} b_1 = a_1, b_2 = 2a_4, b_3 = 3a_9 \text{ etc.)} \]

Prove that

\[
\sum a_n < \infty \iff \sum b_n < \infty.
\]

Hint (which can be avoided): \( n + 1 \leq (n + 1)^2 - n^2 \leq 3n. \)

Write

\[
T_n = \sum_{n^2 \leq k < (n+1)^2} a_k.
\]

Then \( T_n \) is a sum of \( (n + 1)^2 - n^2 \) terms \( a_k \) with

\[
a_{(n+1)^2} \leq a_k \leq a_{n^2}.
\]

Therefore

\[
b_{n+1} \leq T_n \leq 3b_n.
\]

Now since all series have nonnegative terms, \( \sum a_n \) converges if and only if \( \sum a_n \) has bounded partial sums, which happens if and only if \( \sum T_n \) has bounded partial sums, which happens if and only if \( \sum b_n \) has bounded partial sums, which happens if and only if \( \sum b_n \) converges.
3. Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ be continuous. If $X$ is compact, prove $f(X)$ is compact.

Recall that if $U \subset Y$ is open, then $f^{-1}(U) \subset X$ is open, by the continuity of $f$. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of $f(X) \subset Y$. Then $\{f^{-1}(U_\alpha) : \alpha \in A\}$ is an open cover of $X$. Since $X$ is compact, there is a finite set $\alpha_1, \ldots, \alpha_n$ such that

$$X \subset \bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j}).$$

Then

$$f(X) \subset \bigcup_{j=1}^{n} f(f^{-1}(U_{\alpha_j})) \subset \bigcup_{j=1}^{n} U_{\alpha_j}.$$
4. A function \( f : R \to R \) is called BV if there exist monotonic increasing functions \( g \) and \( h \) from \( R \) to \( R \) such that for all \( x \in R \), \( f(x) = g(x) - h(x) \).

(a) Prove that a BV function can have at most a countable set of discontinuities.

If \( x \) is a point of continuity of \( g \) and of \( h \), then \( f \) is a point of continuity of \( f = g - h \). This the set of discontinuities of \( f \) is a subset of the union of the discontinuity sets of \( g \) and \( h \). Since \( g \) and \( h \) are monotone, a theorem from class says that \( g \) and \( h \) have at most countable sets of discontinuities. Since a union of two countable or finite sets is countable or finite, \( f \) has at most a countable set of discontinuities.

(b) Give an example of a function that is not BV.

The characteristic function of the rationals \( Q \) (i.e. \( f(x) = 1 \) if \( x \in Q \) and \( f(x) = 0 \) if \( x \notin Q \)) is not continuous at any point. Since \( R \) is not countable, \( f \) is not BV.