Fall 2006 Math 33B, Lecture 1 Midterm I

Name: 

UCLA ID:

Discussion session (circle one)
1A (Faizal Sainal, Tuesday)  1B (Faizal Sainal, Thursday)
1C (Chris Vogl, Tuesday)  1D (Chris Vogl, Thursday)

Directions: Fill in your name and circle your section above. No outside materials are allowed. Show all the necessary steps involved in finding your solutions, unless otherwise instructed. Good luck.

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1. (25 pts) Suppose a tank initially contains 100 gallons of salt-water solution containing \( k \) lb of salt per gallon. Salt-water solution containing 1 lb of salt for each gallon of water begins entering the tank at a rate of 2 gal/min at \( t = 0 \). Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 1 gal/min. We assume that the tank is of infinite capacity and thus never overflows.

(20 pt)(a) Suppose there is 2 lb. of salt in the tank after 10 minutes. Find \( k \).

Let \( X(t) \) the amount of salt (lb) in the tank at time \( t \). Then we get the DE
\[
\frac{dX}{dt} = \text{rate in} - \text{rate out}
\]
\[
= 2 - \frac{1}{100 + t} X(t).
\]

Multiplying integration factor \( \mu(t) = \exp\int \frac{1}{100+t} dt = 100 + t \), to both sides, one obtains
\[
((100 + t)X(t)) = \int 2(100 + t)dt = 200t + t^2 + C.
\]

Hence \( X(t) = \frac{200t + t^2 + C}{100 + t} \). Since \( X(0) = 100k \), we have
\[
X(t) = \frac{200t + t^2 + (100)^2 k}{100 + t}.
\]

Now if you plug in \( X(10) = 2 \), then we will get negative \( k \), which is not possible physically. Hence you get the full score if you are correct up to this point.

(5 pt) (b) Eventually (as \( t \) goes infinity) how much salt per gallon of water will be in the tank?

As \( t \) goes to infinity, the eventual salt/gallon ratio will be the same as what is coming in, that is 1 lb per gallon.

(We covered a question like this in the class (and in the homework): no more explanation is needed.)
2. (25 pts) For the differential equation

\[(x - y)\frac{dy}{dx} + (f(y) - \cos x) = 0.\]

(15 pt) (a) Suppose the associated differential form is exact with \(f(0) = 0\). Find \(f(y)\) and find the general solution of the form \(F(x, y) = C\).

Here \(P = f(y) - \cos x\) and \(Q = (x - y)\). To be exact, \(dP/dy = f'(y) = dQ/dx = 1\). Since \(f(0) = 0\), we obtain \(f(y) = y\).

We need to look for solutions of \((x - y)dy + (y - \cos x)dx = 0\) in the form \(F(x, y) = C\). We solve for

\[\frac{\partial F}{\partial x} = y - \cos x, \quad \partial F/\partial y = x - y.\]

If we integrate the first equation with respect to \(x\) then we get \(F(x, t) = xy - \sin x + \phi(y)\). If we plug \(F\) into the second equation, we get \(\phi'(y) = -y\). Thus \(\phi(y) = -\frac{1}{2}y^2\) and the solution is

\[F(x, y) = xy - \sin x - \frac{1}{2}y^2 = C.\]

(10 pt) (b) Let \(f(y) = \sin y\). Then the equation is not exact. Show that there is at least one solution of the differential equation with initial value \(y(1) = 0\).

The differential equation in normal form is

\[
\frac{dy}{dx} = \frac{\sin y - \cos x}{x - y}.
\]

The function on the right side is continuous near the point \((x, y) = (1, 0)\). (you may say in a rectangle containing \((1, 0)\), for example \((1/2, 3/2) \times (-1/4, 1/4)\)). Hence by the existence theorem (theorem 7.15) in 2.7, there is at least one solution of the initial value problem.
3. (25 pt) Consider the following initial value problem

\[ y' = y^2 - \cos^2 t - \sin t \text{ and } y(0) = 2. \]

Show that \( y(t) > \cos t \) for all \( t \) for which \( y \) is defined.

First note that \( y = \cos t \) is a solution since

\[ (\cos t)' = -\sin t = (\cos t)^2 - \cos^2 t - \sin t. \]

Also \( f(y, t) = y^2 - \cos^2 t - \sin t \) and \( \frac{\partial f}{\partial y} = 2y \) is continuous for all \((y, t)\), so the uniqueness theorem (theorem 7.16) holds and two solution curves cannot cross or touch each other.

Since \( y(0) = 2 > \cos 0 = 1 \), it follows that \( y(t) > \cos t \) for all \( t \).
4. (25 pt) The graph of the right-hand side of $y' = f(y)$ is given as below:

Identify equilibrium solutions and classify them as stable or asymptotically unstable. Draw solution curves in $t-y$ plane, and explain the behavior of solutions as $t \to \infty$ in terms of their initial condition $y(0) = c$.

$a$ and $c$ are asymptotically unstable and $b$ is stable.

As $t \to \infty$: If $y(0) < a$ then $y(t) \to -\infty$. If $y(0) = a$ then $y = a$. If $a < y(0) < c$, then $y(t) \to b$. If $y(0) \geq c$, then $y(t) \to c$.

(Pictures are omitted).