Problem 1.

(a) Define \( i : \mathcal{F} \to \tilde{\mathcal{F}} \) by

\[
i_U : \mathcal{F}(U) \to \tilde{\mathcal{F}}(U)
\]

\[
a \mapsto \left( i_U(a) : U \to \text{Tot} \tilde{\mathcal{F}} \right)
\]

Remark that \( i_U(a) : U \to \text{Tot} \tilde{\mathcal{F}} \) is a continuous section. Let \( U_1 \subset U_2 \subset X \) be open subsets of \( X \). Consider the below diagram:

\[
\begin{array}{ccc}
\mathcal{F}(U_2) & \xrightarrow{\iota_{U_2}} & \tilde{\mathcal{F}}(U_2) \\
\mathcal{F}(1_{U_2 U_1}) & = & \tilde{\mathcal{F}}(1_{U_2 U_1}) \\
\mathcal{F}(U_1) & \xrightarrow{\iota_{U_1}} & \tilde{\mathcal{F}}(U_1)
\end{array}
\]

Then for \( a \in \mathcal{F}(U_2) \), we have \( \tilde{\mathcal{F}}(1_{U_2 U_1}) \circ i_{U_2}(a)(x) = a_x \) for \( x \in U_1 \). On the other hand, \( i_{U_1} \circ \mathcal{F}(1_{U_2 U_1})(a)(x) = (a|_{U_1})_x \) for \( x \in U_1 \). Indeed, the definition of stalks gives \( a_x = (a|_{U_1})_x \). Hence, the diagram commutes and \( i \) is a natural morphism of presheaves \( i : \mathcal{F} \to \tilde{\mathcal{F}} \).

(b) \( \Leftarrow \) This part is obvious, since \( \tilde{\mathcal{F}} \) is a sheaf.

\( \Rightarrow \) Suppose that \( \mathcal{F} \) is a sheaf. We want to find \( j : \tilde{\mathcal{F}} \to \mathcal{F} \) with \( j \circ i = 1_{\mathcal{F}} \), and \( i \circ j = 1_{\tilde{\mathcal{F}}} \). For any continuous section \( s \) of \( p \) and any open set \( U \subset X \), we have an open cover of \( U \),

\[
(1) \quad U = \bigcup_{a \in \mathcal{F}(U)} U_a
\]

where \( U_a = s^{-1}\{a_x | x \in U\} \). We claim that there exist \( j(s) \in \mathcal{F}(U) \) such that \( j(s)|_{U_a} = a \) for each \( a \in \mathcal{F}(U) \). In fact, \( x \in U_a \cap U_b \) implies \( a_x = b_x \). From SHEAF(2), we obtain \( a|_{U_a \cap U_b} = b|_{U_a \cap U_b} \), and hence we obtain the existence of \( j(s) \) such that \( j(s)|_{U_a} = a \) by SHEAF(3). Furthermore, this \( j(s) \) is uniquely determined by SHEAF(2).

Now, we show that \( j \circ i = 1_{\mathcal{F}} \). Let \( U \subset X \) be open, and \( a \in \mathcal{F}(U) \). From \( i(a)^{-1}\{a_x | x \in U\} = U \), we obtain \( j(i(a)) = a \). It remains to show that \( i \circ j = 1_{\tilde{\mathcal{F}}} \).

We use the open cover of \( U = \bigcup U_a \) again. For \( s \in \tilde{\mathcal{F}}(U) \), \( s(x) = a_x \) for \( x \in U_a \). Definition of \( i \) in (a) implies \( ij(s)(x) = j(s)_x = a_x = s(x) \). Hence, now SHEAF(2) implies that \( ij(s) = s \).
(c) (Existence) We use (b) on $\mathcal{G}$ to obtain an isomorphism $i_G : \mathcal{G} \to \tilde{\mathcal{G}}$, then it suffices to find $j' : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$ such that $i_G^{-1} \circ j' \circ i = j$. Define $j' : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$ by

$$  \tilde{j'}_U : \tilde{\mathcal{F}}(U) \to \tilde{\mathcal{G}}(U) $$

$$  (s : U \to \text{Tot} \tilde{\mathcal{F}}) \mapsto \left( \tilde{j'}_U(s) : U \to \text{Tot} \tilde{\mathcal{G}} \\ x \mapsto [U, j(a)] \in \mathcal{G}_x \right). $$

Then, the diagram

$$ \begin{array}{ccc} \tilde{\mathcal{F}}(U_2) & \xrightarrow{j'} & \tilde{\mathcal{G}}(U_2) \\ \text{res} \downarrow & = & \text{res} \\ \tilde{\mathcal{F}}(U_1) & \xrightarrow{j'} & \tilde{\mathcal{G}}(U_1) \end{array} $$

commutes where $U_1 \subset U_2 \subset X$. Let $\tilde{j} = i_G^{-1} \circ \tilde{j}'$.

(Uniqueness) Let $U \subset X$ be open. We use the open cover in (1) in (b) again, $U = \bigcup U_\alpha$. For any $s \in \tilde{\mathcal{F}}(U)$, we have $i(a|_{U_\alpha}) = s|_{U_\alpha}$. The condition $j \circ i = j$ forces $j(s|_{U_\alpha}) = \tilde{j}(s)|_{U_\alpha} = j(a|_{U_\alpha})$. Hence by SHEAF(2), $\tilde{j}(s)$ is uniquely determined.

**Problem 2.**

(a) SHEAF(1): $\mathcal{G}(\phi) = \{\phi\}$ is a final object in the category SETS.

SHEAF(2): Let $U = \bigcup U_\alpha$ be an open cover. Let $a, b \in \mathcal{G}(U)$ with $a|_{U_\alpha} = b|_{U_\alpha}$ for all $\alpha$. For any $x \in U$, there is some $\alpha$ such that $x \in U_\alpha$. Since $a|_{U_\alpha} = b|_{U_\alpha}$, we have

$$  a(x) = a|_{U_\alpha}(x) = b|_{U_\alpha}(x) = b(x). $$

Hence, $a = b$.

SHEAF(3): Let $U = \bigcup U_\alpha$ be an open cover. Let $a_\alpha \in \mathcal{G}(U_\alpha)$ satisfy

$$  a_\alpha|_{U_\alpha \cap U_\beta} = a_\beta|_{U_\alpha \cap U_\beta} $$

for any $\alpha, \beta$. We define $a \in \mathcal{G}(U)$ by

$$  a(x) = a_\alpha(x) $$

where $x \in U_\alpha$. Then we have $a|_{U_\alpha} = a_\alpha$.

(b) The natural morphism $i : \mathcal{F} \to \mathcal{G}$ is defined by

$$  i_U : \mathcal{F}(U) = A \to \mathcal{G}(U) $$

$$  a \mapsto \left( i_U(a) : U \to A \\ x \mapsto a \right). $$

Recall from Problem 1 (a) that $i : \mathcal{F} \to \tilde{\mathcal{F}}$ is defined by

$$  i_U : \mathcal{F}(U) \to \tilde{\mathcal{F}}(U) $$

$$  a \mapsto \left( i_U(a) : U \to \text{Tot} \tilde{\mathcal{F}} \\ x \mapsto a_x \right). $$
We show that these morphisms are isomorphic by showing that
\[ f : \mathcal{G}(U) \longrightarrow \mathcal{F}(U) \]
\[ (s : U \rightarrow A) \mapsto (\tilde{s} : U \rightarrow \text{Tot } \mathcal{F}) \]
is an isomorphism of sets.

(Case 1) Suppose \( s_1(x) \neq s_2(x) \) for some \( x \in U \), then \( s_1(x)|_x \neq s_2(x)|_x \). Thus, \( f(s_1) \neq f(s_2) \).

(Case 2) Let \( \tilde{s} : U \rightarrow \text{Tot } \mathcal{F} \) be a continuous section. For each \( x \in U \), define \( s(x) \in A \) by \( \tilde{s}(x) = s(x)|_x \). Then, for fixed \( a \in A \), we have
\[ \{ x \in U | s(x) = a \} = \{ x \in U | \tilde{s}(x) \in \{ a_x | x \in U \} \}. \]

The RHS is an open set since \( \tilde{s} \) is continuous, thus LHS is also an open set in \( U \). Since this is true for all \( a \in A \), we conclude that \( s \) is continuous and \( f(s) = \tilde{s} \). Hence the natural morphism \( i : \mathcal{F} \rightarrow \mathcal{G} \) is isomorphic to \( i : \mathcal{F} \rightarrow \mathcal{F} \).

**Problem 3.**

Let \( P \) be a nonzero prime ideal in \( A = \mathbb{Z}[X] \). Then the natural homomorphism \( \mathbb{Z} \longrightarrow A/P \) has kernel \( P \cap \mathbb{Z} \). This gives an embedding of \( \mathbb{Z}/(P \cap \mathbb{Z}) \) into \( A/P \).

Since \( A/P \) is an integral domain, so is \( \mathbb{Z}/(P \cap \mathbb{Z}) \). Thus, we have two cases
\[ P \cap \mathbb{Z} = \begin{cases} p\mathbb{Z} & \text{for some prime } p \in \mathbb{Z} \\ (0) & \end{cases} \]

(Case 1) \( P \cap \mathbb{Z} = p\mathbb{Z} \) for some prime \( p \in \mathbb{Z} \):
By 3rd isomorphism theorem, we have
\[ A/P \simeq (\mathbb{Z}/p\mathbb{Z}[X])/((P/p\mathbb{Z}[X] \cap \mathbb{Z})). \]

In fact \( \mathbb{Z}/p\mathbb{Z}[X] = \mathbb{F}_p[X] \), and the LHS is an integral domain. It follows that \( P/p\mathbb{Z}[X] \) is a prime ideal in \( \mathbb{F}_p[X] \). Since \( \mathbb{F}_p[X] \) is UFD, \( P/p\mathbb{Z}[X] = (f(X)) \) for some \( f(X) \in \mathbb{F}_p[X] \) irreducible polynomial of degree \( \geq 1 \) or \( P/p\mathbb{Z}[X] = (0) \). Hence, in this case, we obtain \( P = (p, f(X)) \) or \( P = p\mathbb{Z}[X] \) where \( f \) is irreducible mod \( p \).

(Case 2) \( P \cap \mathbb{Z} = (0) \):
Consider the ideal \( P\mathbb{Q}[X] \subset \mathbb{Q}[X] \), this is a proper prime ideal in \( \mathbb{Q}[X] \). So, \( P\mathbb{Q}[X] = f(X)\mathbb{Q}[X] \) where \( f \) is irreducible over \( \mathbb{Q} \). Further, we can assume that the polynomial \( f \) is primitive. We claim that \( P = f(X)\mathbb{Z}[X] \). Suppose \( h \in P \), \( h = fg \) for some \( g \in \mathbb{Q}[X] \). Taking content(Gauss lemma) on each side, we obtain \( g \in \mathbb{Z}[X] \). Hence it follows that \( P = f(X)\mathbb{Z}[X] \).

Now, we can write the result as follows:
Prime ideals \( P \) in \( \mathbb{Z}[X] \) are of the following forms:
\[
(2) \quad P = \begin{cases} (0), & \text{for } f \in \mathbb{Z}[X] \text{ irreducible and primitive}, \\ (p), & \text{for some prime } p \in \mathbb{Z}, \\ (p, f(X)), & \text{for some prime } p \in \mathbb{Z}, \text{ and } f \text{ is irreducible mod } p. \end{cases}
\]

Now, we characterize the topology on \( \text{Spec}(\mathbb{Z}[X]) \). Let \( I \subset \mathbb{Z}[X] \) be a proper ideal. Consider \( I \cap \mathbb{Z} = n\mathbb{Z} \), we have two cases,
(Case 1) \( I \cap \mathbb{Z} = n\mathbb{Z} \) with \( n \neq 0, \pm 1 \):
Let \( p \in V(I) \), i.e. \( p \) is a prime ideal containing \( I \). Then \( p \cap \mathbb{Z} = p\mathbb{Z} \) for some prime \( p \nmid n \). Fix a prime \( p \nmid n \). The ideal \( I + p\mathbb{Z}[X] \subset \mathbb{Z}[X] \) maps to some ideal \( (f(X)) \subset \mathbb{F}_p[X] \) by reducing mod \( p \), since \( \mathbb{F}_p[X] \) is a PID. Let \( f_i \) be distinct irreducible factors of \( f \) in \( \mathbb{F}_p[X] \) if \( \deg(f) > 0 \), and enumeration of all irreducible polynomials of \( \mathbb{F}_p[X] \) with 0 if \( f = 0 \). Thus, we have \( (f(X)) \subset (f_i(X)) \subset \mathbb{F}_p[X] \) for each \( i \). Pulling back these ideals to \( \mathbb{Z}[X] \), we obtain \( I \subset I + p\mathbb{Z}[X] \subset (p, f_i(X)) \subset \mathbb{Z}[X] \) for each \( i \). Hence, the result

\[
V(I) = \{(p, f_{p,i}) \mid p|\n, (I + p\mathbb{Z}[X])/(p\mathbb{Z}[X]) = (f_p(X)) < \mathbb{F}_p[X], \deg(f_p) > 0, f_{p,i} \text{ are distinct irreducible factor of } f \text{ in } \mathbb{F}_p[X]\}
\]

\[
\bigcup\{(p, f_{p,i}) \mid p|\n, (I + p\mathbb{Z}[X])/(p\mathbb{Z}[X]) = (0) < \mathbb{F}_p[X], f_{p,i} \text{ are enumeration of all irreducible polynomials of } \mathbb{F}_p[X] \text{ with } 0\}.
\]

(Case 2) \( I \cap \mathbb{Z} = (0) \):
Consider \( I \mathbb{Q}[X] = (f(X)) \subseteq \mathbb{Q}[X] \) with \( f \) being primitive. Then we obtain \( I = f(X)\mathbb{Z}[X] \) by Gauss lemma. Let \( f_i \) be distinct irreducible factors of \( f \), and \( \tau_i \in \mathbb{C} \) be the corresponding roots of \( f_i \). For each \( i \), we need to find primes \( p \) such that \( (p, f_i) \) become proper. To do this, we use Gauss lemma again so that we obtain the result:

\( (p, f_i) \) is proper \( \iff \frac{1}{p} \notin \mathbb{Z}[\tau_i] \).

Hence, we have,

\[
V(I) = \{(p, f_{i,j}) \mid f_i|f, \frac{1}{p} \notin \mathbb{Z}[\tau_i], ((f_i) + p\mathbb{Z}[X])/(p\mathbb{Z}[X]) = (f_i(X) \text{ mod } p) < \mathbb{F}_p[X], \deg(f_i \text{ mod } p) > 0, f_{i,j} \text{ are distinct irreducible factor of } f_i \text{ in } \mathbb{F}_p[X]\}
\]

\[
\bigcup\{(f_i) | f_i|f \text{ irreducible}\}.
\]

**Problem 4.**
Let \( X = Y = \text{Spec} \mathbb{C}, \ S = \text{Spec} \mathbb{R} \). They are all affine schemes. Also, \( \mathbb{C} \) can be regarded as \( \mathbb{R} \)-algebra. Then \( Z = \text{Spec}(\mathbb{C} \otimes_\mathbb{R} \mathbb{C}) \) is the desired pull-back of the diagram,

\[
\begin{array}{ccc}
Z & \longrightarrow & \text{Spec} \mathbb{C} \\
\downarrow & & \downarrow \text{Spec} \mathbb{C} \\
\text{Spec} \mathbb{C} & \longrightarrow & \text{Spec} \mathbb{R}
\end{array}
\]

Since \( \mathbb{C} \otimes_\mathbb{R} \mathbb{C} \) is isomorphic to \( \mathbb{C} \times \mathbb{C} \) as rings, we have \( Z = \text{Spec}(\mathbb{C} \times \mathbb{C}) \). Hence, we obtain the result

\( Z = \text{Spec} \mathbb{C} \times_{\text{Spec} \mathbb{R}} \text{Spec} \mathbb{C} = \{\mathbb{C} \times \{0\}, \{0\} \times \mathbb{C}\} \).

**Problem 6.**
\( \iff \) Suppose first that \( A \) has a nontrivial idempotent \( a \). Then, we claim that \( A = aA \oplus (1 - a)A \). For any \( x \in A \), \( x = ax + (1 - a)x \), so \( A = aA + (1 - a)A \). If \( y \in aA \cap (1 - a)A \), then \( ay \in a(1 - a)A = 0 \), and \( (1 - a)y \in (1 - a)aA = 0 \). Thus, \( y = 0 \) and \( A = aA \oplus (1 - a)A \).
Now, we have \( V(aA) \cap V((1 - a)A) = V(aA + (1 - a)A) = V(A) = \emptyset \), and \( V(aA) \cup V((1 - a)A) = V(aA \cap (1 - a)A) = V((0)) = \text{Spec}(A) \). Hence, \( \text{Spec}(A) \) is covered by disjoint union of nonempty closed sets \( V(aA) \) and \( V((1 - a)A) \), i.e. \( \text{Spec}(A) \) is disconnected.

\[ \Rightarrow \text{ Suppose that } \text{Spec}(A) \text{ is disconnected, i.e. } \text{Spec}(A) = V(J_1) \cup V(J_2), \text{ with } (J_1) \cap (J_2) = \emptyset \text{ for some ideals } J_1, J_2 \subset A, \text{ and } V(J_1) \neq \emptyset, V(J_2) \neq \emptyset. \text{ If } I, J \text{ are radical ideals, then we have } V(I) = V(J) \iff I = J. \text{ Take } I_1 = \sqrt{J_1} \text{ and } I_2 = \sqrt{J_2}. \text{ Then, we obtain } I_1 + I_2 = A \text{ and } I_1 \cap I_2 = \sqrt{(0)}. \text{ Also, we know that } I_1, I_2 \text{ are proper. Thus, we can find } a \in I_1, b \in I_2 \text{ such that } a + b = 1. \text{ However } ab \in I_1I_2 = \sqrt{(0)}, \text{ so we see that } (ab)^n = 0 \text{ for some } n \geq 1. \text{ Using } (a + b)^{2n} = 1, \text{ we obtain } a^nA + b^nA = A. \text{ Let } a' \in a^nA, \text{ and } b' \in b^nA \text{ with } a' + b' = 1. \text{ Then, } a' \text{ is the desired nontrivial idempotent, since } a' = a'(a' + b') = a'^2 + a'b' = a'^2 \text{ implies } a'^2 = a'. \]

**Problem 7.**

Remark that \( f(P) \) is the image of \( f \) in the residue field \( A_P/(PA_P) \) where \( A_P \) is the localization. We claim that \( f(P) = 0 \) if and only if \( f \in P \). We see that

\[
f(P) = 0 \iff \text{There exists } s \in S = A - P \text{ such that } fs/s \in PA_P
\]

\[
\iff \text{There exists } s', s'' \in S \text{ and } p \in P \text{ such that } (p - fs')ss'' = 0
\]

\[
\iff f \in P
\]

, since \( s's'' \notin P \). Hence, the set \( \{ P \in \text{Spec}(A) | f(P) = 0 \} \) is just \( V(fA) \), so it is closed.

**Problem 8.**

Take \( A = \mathbb{Z}[X] \), and \( U = D(2) \cup D(x) = \text{Spec}(A) - (V(2A) \cap V(xA)) \). Suppose \( U = D(a) \) for some \( a \in A \). Then,

\[
U = D(a) \iff \text{Spec}(A) - V(aA) = \text{Spec}(A) - V((2, x))
\]

\[
\iff V(aA) = V((2, x)) = \{(2, x)\}.
\]

, since \( (2, x) \) is maximal ideal. Thus, for a prime ideal \( p \in \text{Spec}(A) \), we have \( p \supseteq aA \iff p = (2, x) \). Suppose \( \deg(a) > 0 \), then we can find an irreducible factor \( b \in A \) of \( a \). Further, as in problem 3, we can find a prime number \( p \in \mathbb{Z} \) such that \( (p, b) \) is proper. Then, we obtain \( V(aA) \supseteq \{(b), m\} \), where \( m \) is a maximal ideal that contains \( (p, b) \), and this is a contradiction to \( V(aA) = \{(2, x)\} \). Now, we assume that \( a \in \mathbb{Z} \). Our assumption implies that \( a \) cannot be unit or zero. Then, there is a prime number \( p \in \mathbb{Z} \) such that \( V(aA) \supseteq \{(p), (p, x)\} \). This again contradicts \( V(aA) = \{(2, x)\} \). Hence \( U = D(a) \) is impossible for any \( a \in A \).

**Problem 9.**

First, consider \( Mor_{\text{Rings}}(\mathbb{Z}, \mathbb{Q}) = \{ i : \mathbb{Z} \hookrightarrow \mathbb{Q} \} \), and \( Mor_{\text{TopSpaces}}(\text{Spec} \mathbb{Q}, \text{Spec} \mathbb{Z}) = \{ f : \text{Spec} \mathbb{Q} \rightarrow \text{Spec} \mathbb{Z} | f \text{ is continuous} \} \). We have only one point in \( Mor_{\text{Rings}}(\mathbb{Z}, \mathbb{Q}) \), but \( Mor_{\text{TopSpaces}}(\text{Spec} \mathbb{Q}, \text{Spec} \mathbb{Z}) \) contains infinitely many points, since it contains \( f_p : (0) \subset \mathbb{Q} \mapsto p\mathbb{Z} \subset \mathbb{Z} \) for all prime \( p \in \mathbb{Z} \). Thus, the functor \( \text{Spec}(-) \) is not full.

Then we consider \( Mor_{\text{Rings}}(\mathbb{C}, \mathbb{C}) \supseteq \{ i, c \} \), where \( i \) is identity, \( c \) is complex conjugation. Also, consider \( Mor_{\text{TopSpaces}}(\text{Spec} \mathbb{C}, \text{Spec} \mathbb{C}) = \{ i_0 \} \), where \( i_0 : (0) \subset \mathbb{C} \mapsto (0) \subset \mathbb{C} \). We have at least two points in \( Mor_{\text{Rings}}(\mathbb{C}, \mathbb{C}) \), but we have only one
Problem 12.

SHEAF (1): \( F(\emptyset) = \{ \emptyset \} \) is a final object in the category SETS.

SHEAF (2): Let \( U = \bigcup U_\alpha \) be an open cover. Let \( a, b \in F(U) \) with \( a|_{U_\alpha} = b|_{U_\alpha} \) for all \( \alpha \). For any \( x \in U \), there is some \( \alpha \) such that \( x \in U_\alpha \). Since \( a|_{U_\alpha} = b|_{U_\alpha} \), we have
\[
a(x) = a|_{U_\alpha}(x) = b|_{U_\alpha}(x) = b(x).
\]

Hence, \( a = b \).

SHEAF (3): Let \( U = \bigcup U_\alpha \) be an open cover. Let \( a_\alpha \in F(U_\alpha) \) satisfy
\[
a_\alpha|_{U_\alpha \cap U_\beta} = a_\beta|_{U_\alpha \cap U_\beta}
\]
for any \( \alpha, \beta \). We define \( a \in F(U) \) by
\[
a(x) = a_\alpha(x)
\]
where \( x \in U_\alpha \). Then we have \( a|_{U_\alpha} = a_\alpha \).

Thus, \( F \) is a sheaf.

We claim that \( (X, F) \) is a scheme. First, consider \( F_x = \{ [U, a]|x \in U \in \text{open } X, a \in F(U) \} \). Since \( X \) is a discrete topological space, we can further show that \( F_x = \{ \{ \{ x \}, a \}|a \in F(\{ x \}) \} \simeq k \). This shows that \( (X, F) \) is a local ringed space. For any \( x \in X \), \( U = \{ x \} \), we have \( \{ \{ x \}, F(\{ x \}) \} \simeq (\text{Spec} k) \). This proves our claim.

Suppose that \( (X, F) \simeq (\text{Spec} A, O_A) \). For any \( p \in \text{Spec} A \), \( (O_A)_p = A_p = k \). Since \( A_p \) is a local ring with a unique maximal ideal \( pA_p \) and \( k \) is a field, we must have \( p = 0 \). Thus, \( \text{Spec} A = \{ 0 \} \), and \( A \) cannot have nonzero unit, otherwise \( \text{Spec} A \) would contain nonzero maximal ideal of \( A \). It follows that \( A \) is a field, and \( (X, F) \) is affine if and only if \( X \) is a singleton set.

Problem 13.

(a) We remark that for any \( f \in K \), there is \( n \in \mathbb{N} \) such that \( f \in k(X_1, \ldots, X_n) \). So, there is \( n \in \mathbb{N} \) such that \( (a_1, a_2, \ldots, a_i) = 0 \) for \( i \geq n \). Also, for any \( f, g \in K - \{ 0 \} \), we have \( a(fg) = a(f) + a(g), a(f + g) \geq \min(a(f), a(g)) \), where the addition is componentwise. Then, it follows that \( \{ f \in K|a(f) = 0 \} \) is the set of units in \( A \) and \( \{ f \in K|a(f) > 0 \} \) forms the ideal of all nonunits in \( A \). Further, we obtain that if \( a(f) = (a_1, a_2, \ldots, a_n, 0, 0, \ldots) \), then \( f = uX_1^{a_1}X_2^{a_2} \cdots X_n^{a_n} \) for some unit \( u \in A \). Now, we claim that \( Q := \sum_{i \in \mathbb{N}} X_i A = \{ f \in K|a(f) > 0 \} \). The inclusion \( \subseteq \) is clear. To prove \( \supseteq \), let \( f = X_1 f_2 + \cdots + X_m f_m \). Then, \( a(f) \geq \max(a(X_1 f_2), \ldots, a(X_m f_m)) > 0 \). Hence, the claim is proved and \( Q \) is the unique maximal ideal of \( A \).

(b) The inclusion \( P_i \subset P_{i+1} \) is clear for all \( i \geq 1 \). Also, \( P_i \subset Q \) is obvious, since \( Q = \{ f \in A|a(f) > 0 \} \). To show that \( P_i \) is a prime ideal in \( A \), let \( f, g \in A - P_i \). Then, for some \( b(f)_{i+1}, b(f)_{i+2}, \ldots \) and \( b(g)_{i+1}, b(g)_{i+2}, \ldots \), we have \( a(f) \leq (0, \ldots, 0, b(f)_{i+1}, b(f)_{i+2}, \ldots) \), and \( a(g) \leq 0, \ldots, 0, b(g)_{i+1}, b(g)_{i+2}, \ldots) \). Adding these up, we obtain
\[
a(fg) = a(f) + a(g) \leq (0, \ldots, 0, b(f)_{i+1} + b(g)_{i+1}, b(f)_{i+2} + b(g)_{i+2}, \ldots).
\]
This implies \( fg \in A - P_i \). Hence, \( P_i \) is a prime ideal in \( A \). Furthermore, the same argument as in (a) shows that \( P_i = \sum_{j \leq i} X_j A \).
(c) Let $P$ be a prime ideal in $A$. Let $f \in P - \{0\}$, and $f = uX^a_1 \cdots X^a_n$ for some unit $u \in A$. Since $P$ is a prime ideal, we can find $i \leq n$ such that $X_i \in P$. Define a set $B = \{i \in \mathbb{N} | X_i \in P\}$. We divide into two cases:

(Case1) $B$ is finite:
We can show that $P$ contains all $X_n$ from the formula(*),
$$X_i = X_j \frac{X_i}{X_j} - X_j$$
for $i < j$. Hence, we obtain $P = Q$.

(Case2) $B$ is finite:
By the formula(*), there exists $n \in N$ such that $B = \{i \in \mathbb{N} | 1 \leq i \leq n\}$. Hence, it follows that $P = \sum_{i \leq n} X_i A = P_n$.

We proved that $\text{Spec}A = \{0, Q, P_1, P_2, \cdots, P_n, \cdots\}$.
The Zariski topology $T$ on $\text{Spec}A$ is $T = \{\emptyset\} \cup \{V(p)|p \in \text{Spec}A\}$. To prove this, let $I$ be a proper ideal in $A$. Then, consider $m = \min\{n \in \mathbb{N} | I \subset P_n\}$. If $m \in \mathbb{N}$, then, $V(I) = V(P_m)$. If $m = \infty$, then $V(I) = V(Q) = \{Q\}$. In fact, $V(P_n) = \{Q, P_n, P_{n+1}, \cdots\}$ for each $n \geq 1$.

(d) The topology on $\text{Spec}A - \{Q\}$ is the subspace topology
$$T' = \{\emptyset\} \cup \{V(p) - \{Q\}|p \in \text{Spec}A\}.$$ For each point $P_n \in \text{Spec}A - \{Q\}$, we have $\{P_n\} = V(P_n) - \{Q\} \neq \{P_n\}$. For $0 \in \text{Spec}A - \{Q\}$, $\{0\} = V(0) - \{Q\} \neq \{0\}$. Hence, the scheme $\text{Spec}A - \{Q\}$ has no closed points.

Problem 14.
As a map of topological spaces, it is clear that $X \xymatrix{\rightarrow\ar{f}} Y$ factors through $X \xymatrix{\rightarrow\ar{f}} U \xymatrix{\rightarrow\ar{g}} Y$, where $U \xymatrix{\rightarrow\ar{}\ar{\subseteq}} Y$ is the inclusion. Let $f(x) = y$, we have composition of morphisms of schemes $(X, \mathcal{F}) \xymatrix{\rightarrow\ar{}\ar{f}} (U, \mathcal{G}|_U) \xymatrix{\rightarrow\ar{}\ar{g}} (Y, \mathcal{G})$. This induces morphisms of local rings $\mathcal{G}_y \xymatrix{\rightarrow\ar{}\ar{id}} (\mathcal{G}|_U)_y \xymatrix{\rightarrow\ar{}\ar{g}} \mathcal{F}_x$, since $y \in U$. Further, we know that the ring homomorphism $\mathcal{G}_y \xymatrix{\rightarrow\ar{}\ar{id}} \mathcal{F}_x$ is a local. Thus, $\mathcal{G}_y \xymatrix{\rightarrow\ar{}\ar{id}} (\mathcal{G}|_U)_y \xymatrix{\rightarrow\ar{}\ar{g}} \mathcal{F}_x$ is a composition of local ring homomorphisms. Hence the morphism of schemes $f$ factors through $X \xymatrix{\rightarrow\ar{g}} U$ and $U \xymatrix{\rightarrow\ar{\subseteq}} Y$.

Problem 16.
Let $(X, \mathcal{O}_X) \xymatrix{\rightarrow\ar{f}} (\text{Spec} \mathbb{Z}, \mathcal{O}_Z)$ be a morphism of schemes, and let $f(x) = y$. Then, we have a local ring homomorphism $\mathcal{O}_{Z,y} \xymatrix{\rightarrow\ar{g}} \mathcal{O}_{X,x}$. We have two cases, (Case1) $y = 0$:
Since $\mathcal{O}_{Z,0} = \mathbb{Z}_{(0)} \xymatrix{\rightarrow\ar{g}} \mathcal{O}_{X,x}$ is local, $\mathcal{O}_{Z,0} = \mathbb{Q} \xymatrix{\rightarrow\ar{g}} \mathcal{O}_{X,x}/\mathfrak{m} \xymatrix{\rightarrow\ar{g}} k(x)$ where $\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}_{X,x}$. Thus, characteristic of $k(x)$ is 0.

(Case2) $y = p\mathbb{Z}$:
$\mathcal{O}_{Z,p\mathbb{Z}} = \mathcal{O}_{Z,(p)} \xymatrix{\rightarrow\ar{g}} \mathcal{O}_{X,x}$. Since the homomorphism is local, we have $\mathcal{O}_{Z,(p)}/p\mathcal{O}_{Z,(p)} \xymatrix{\rightarrow\ar{g}} \mathcal{O}_{X,x}/p\mathfrak{m} \xymatrix{\rightarrow\ar{g}} k(x)$. Since $\mathcal{O}_{Z,(p)}/p\mathcal{O}_{Z,(p)} = \mathcal{O}_{Z,p\mathbb{Z}}$, we have $\mathcal{O}_{Z,p\mathbb{Z}} \xymatrix{\rightarrow\ar{g}} \mathcal{O}_{X,x}/p\mathfrak{m} \xymatrix{\rightarrow\ar{g}} k(x)$. Thus, characteristic of $k(x)$ is $p$.
Hence, in either case, we have $f(x) = p\mathbb{Z}$, where $p$ is the characteristic of the residue field $k(x)$.

Problem 19.
We remark that the scheme structure on \( \text{Proj} S \) is given as follows. For each \( p \in \text{Proj} S \), we consider the ring \( S(p) \) of degree zero in the localized ring \( T^{-1} S \), where \( T \) is the multiplicative system consisting of all homogeneous elements of \( S \) which are not in \( p \). For any open subset \( U \subset \text{Proj} S \), we define \( \mathcal{O}(U) \) to be the set of functions \( s : U \rightarrow \coprod S(p) \) such that for each \( p \in U \), \( s(p) \in S(p) \), and such that: for each \( p \in U \), there exists a neighborhood \( V \) of \( p \) in \( U \), and homogeneous elements \( a, f \) in \( S \) of the same degree, such that for all \( q \in V \), \( f q \neq q \), and \( s(q) = a/f \) in \( S(q) \). This \( \mathcal{O} \) is a sheaf and (\( \text{Proj} S, \mathcal{O} \)) is a scheme. Furthermore, for any \( p \in \text{Proj} S \), the stalk \( \mathcal{O}_p \) is isomorphic to the local ring \( S(p) \).

We define \( f : (\text{Proj} S, \mathcal{O}) \rightarrow (\text{Spec} S_0, \mathcal{O}_{S_0}) \) by \( p \in \text{Proj} S \mapsto f(p) = q = p \cap S_0 \), and for basic open set \( D(a) \subset \text{Spec} S_0 \), define \( f^p_{D(a)} : \mathcal{O}_{S_0}(D(a)) = (S_0)_a \rightarrow \mathcal{O}(f^{-1}(D(a))) \) by:

\[
f^p_{D(a)} : (S_0)_a \rightarrow \mathcal{O}(f^{-1}(D(a)))
\]

\[
b/a^m \mapsto \left( f^p_{D(a)}(b/a^m) : f^{-1}(D(a)) \rightarrow \coprod S(p) \right)_{q \mapsto b/a^m}.
\]

This \( f^p \) induces a ring homomorphism of stalks (local rings):

\[
(\mathcal{O}_{S_0})_q = (S_0)_q \xrightarrow{f^p_q} \mathcal{O}_p = S(p)
\]

\[
b/f \mapsto b/f.
\]

Since, \( q = p \cap S_0 \), this \( f^p \) is a local ring homomorphism. Hence, we conclude that \( (f, f^p) : (\text{Proj} S, \mathcal{O}) \rightarrow (\text{Spec} S_0, \mathcal{O}_{S_0}) \) is a natural morphism of schemes.

**Problem 20.**
Let \( I \subset A \) be a homogeneous ideal, and let \( I_d \subset I \) be the set of all homogeneous elements in \( I \) having degree \( d \). Then, we have

\[
I = \bigoplus_{d \geq 0} I_d.
\]

Furthermore, any ideal \( I \) satisfying this property is homogeneous. To show that \( I \) is homogeneous, consider \( a \in I \). By definition of \( I \), there is \( m \geq 0 \) such that \( a t_i^m \in I \) for all \( 0 \leq i \leq n \). Write \( a = \sum_{d \geq 0} a_d \), where \( a_d \) is homogeneous element of degree \( d \). Then, \( a_d t_i^m \in I \) since \( a t_i^m = \sum_{d \geq 0} a_d t_i^m \in I \) with \( a_d t_i^m \) having degree \( d + n \), homogeneous. Since this holds for every \( 0 \leq i \leq n \), we see that \( a_d \in I \) for every \( d \geq 0 \). Hence, \( I \) is homogeneous.

**Problem 21.**
Define \( V(I) = \{ p \in \text{Proj} A | I \subset p \} \).

\( \Rightarrow \) It suffices to show that \( I \) and \( \bar{I} \) define the same closed subschemes of \( \text{Proj} A \).
First, we show the set-theoretic equality \( V(I) = V(\bar{I}) \). It is clear that \( V(I) \supset V(\bar{I}) \), since \( I \subset \bar{I} \). Let \( p \in V(I) \) and \( a \in \bar{I} \). We claim that \( a \in p \). By definition of \( \bar{I} \), we have some \( m \geq 0 \) such that \( a t_i^m \in I \) for all \( 0 \leq i \leq n \). Suppose \( a \notin p \), then we must have \( t_i^m \notin p \) for all \( i \). This forces that the ideal \( (t_0, \cdots, t_n) \) is contained in \( p \). Since \( p \in \text{Proj} A \), \( p \) cannot contain \( (t_0, \cdots, t_n) \). Thus, we proved our claim, namely, \( V(I) \subset V(\bar{I}) \). Hence, \( V(I) = V(\bar{I}) \) follows.
Now, we need to show that $V(I) \simeq \text{Proj}(A/I)$ and $V(\overline{I}) \simeq \text{Proj}(A/\overline{I})$ are isomorphic as schemes. For, we consider the canonical surjection $A/I \longrightarrow A/\overline{I}$ given by $\widetilde{a} \mapsto \widehat{a}$. This induces a surjection of localized rings $(A/I)_{(f)} \longrightarrow (A/\overline{I})_{(f)}$ which associates $\overline{a}/f^r$ to $\widehat{a}/f^r$ for homogeneous $a, f$ with $\deg f > 0$ and $\deg a = r\deg f$. It will be enough to show that this map is also injective. But if $\overline{a}/f^r = 0$, then $f^m a \in \overline{I}$. There is an integer $N$ such that $t_0^N f^m a, \ldots, t_N^N f^m a \in I$ and for $k$ large enough $f^k a \in I$, so $\overline{a}/f^r = 0$ in $(A/I)_{(f)}$.

$\Leftarrow$) Suppose $I, J$ define the same closed subschemes of $\text{Proj} A$. Then, we have $(A/I)_{(f)} \simeq (A/J)_{(f)}$ for any homogeneous element $f \in A^+$ via $\overline{a}/f^r \mapsto \widehat{a}/f^r$. By the way, $a \in J$ if and only if there is $m$ such that $at_i^m \in J$ for all $i$. This means that $\overline{a} = 0$ in $(A/J)_{(t_i)}$ for each $i$. By the isomorphism, we have $\widehat{a} = 0$ in $(A/I)_{(t_i)}$ for each $i$. Again, this is equivalent to $at_i^m \in I$ for some $m_i$. Taking maximum of $m_i$, we obtain that $a \in \overline{I}$. Hence, $a \in J \iff a \in \overline{I}$, giving that $I = \overline{J}$. 
