1.1 Let $A = k[[T]]$ be the ring of formal power series with coefficients in a field $k$. Determine $\text{Spec} A$.

Proof. We begin with a claim that $A^\times = \{\sum a_i T^i \in A : a_i \in k$, and $a_0 \in k^\times\}$. Let $\sum a_i T^i \in A^\times$, then there exists $\sum b_i T^i \in A$ such that $(\sum a_i T^i)(\sum b_i T^i) = 1$. The constant term in both side should agree, so we have $a_0 b_0 = 1$, giving that $a_0 \in k^\times$. Conversely, let $a_0 \in k^\times$. By multiplying $a_0^{-1}$, we can assume that $a_0 = 1$. Let $g = -\sum_{i \geq 1} a_i T^i$, then we have $\sum a_i T^i = 1 - g$. The formal power series for $\sum_{i \geq 0} g^i$ is the desired inverse for $\sum a_i T^i$. Thus, our claim is proved.

Let $P \subset A$ be a nonzero prime ideal in $A$. Denote the number $\min\{n \in \mathbb{N} : T^n \in P\}$ by $n(P)$. By our previous claim, we have $n(P) \geq 1$, and we also have $T^n(P) \in P$. $n(P) \geq 2$ cannot happen because $P$ is a prime ideal. Thus, we must have $n(P) = 1$, i.e. $T \in P$. It follows that $P = (T)$. Hence $\text{Spec} A = \{(0), (T)\}$. □

1.2 Let $\varphi : A \rightarrow B$ be a homomorphism of finitely generated algebras over a field $k$. Show that the image of a closed point under $\text{Spec} \varphi$ is a closed point.

Proof. Let $m$ be a maximal ideal in $B$, it suffices to show that $\text{Spec} \varphi(m) = \varphi^{-1}(m)$ is a maximal ideal in $A$. By 1st isomorphism theorem, $\varphi$ induces an embedding $A/\varphi^{-1}(m) \hookrightarrow B/m$. Then Weak Nullstellensatz shows that $B/m$ is a finite field extension of $k$. It follows that $A/\varphi^{-1}(m)$ is an integral domain which is finite dimensional $k$-vector space. For any $a \in (A/\varphi^{-1}(m)) \setminus \{0\}$, the set $\{1, a, a^2, \cdots\}$ is linearly dependent over $k$. Thus, $a$ satisfies a polynomial equation $\sum_{i \leq N} c_i a^i = 0$ with $c_i \in k$ and $c_0 \neq 0$. This implies that $a \sum_{i \geq 1, N} c_i a^{-1} = -c_0 \in k^\times$. Hence $a$ is invertible in $A/\varphi^{-1}(m)$, giving that $A/\varphi^{-1}(m)$ is a field. □

1.3 Let $k = \mathbb{R}$ be the field of real numbers. Let $A = k[X,Y]/(X^2 + Y^2 + 1)$. We wish to describe $\text{Spec} A$. Let $x, y$ be the respective images of $X, Y$ in $A$.

(a) Let $m$ be a maximal ideal of $A$. Show that there exist $a, b, c, d \in k$ such that $x^2 + ax + b, y^2 + cy + d \in m$. Using the relation $x^2 + y^2 + 1 = 0$, show that $m$ contains an element $f = ox + by + \gamma$ with $(\alpha, \beta) \neq (0, 0)$. Deduce from this that $m = fA$.

Proof. Note that $A/m = C$. This is because it is a finite extension of $k$ by Weak Nullstellensatz, and it contain $x, y$ with $x^2 + y^2 + 1 = 0$. Thus, it is clear that $x, y$ satisfy some quadratic equation over $k$. We add up those quadratic equations to obtain

$$ax + cy + b + d - 1 = 0 \in A/m.$$ 

(Case 1) $(a, c) \neq (0, 0)$: We may take $f = ax + cy + b + d - 1 \in m$.

(Case 2) $a = c = 0$, and hence $b + d = 1$: Either $x, y \in A/m$ are both pure imaginary, or one of them is real. Note that $x, y$ are not both 0. Thus, we can find $\alpha, \beta, \gamma$ such that $\alpha x + \beta y + \gamma = 0 \in A/m$. 

1
We now consider $A/fA$, assume that $f = \alpha x + \beta y + \gamma$ with $\alpha \neq 0$. Then $A/fA$ becomes

\[ k[X,Y]/(X^2 + Y^2 + 1, f) = k[X]/(X^2 + ((-\beta/\alpha)x - \gamma/\alpha)^2 + 1) = \mathbb{C}. \]

Therefore, $fA$ is a maximal ideal in $A$, and combined with $fA \subset m$, we have $m = fA$. \hfill \square

(b) Show that the map $(\alpha, \beta, \gamma) \mapsto (\alpha x + \beta y + \gamma)A$ establishes a bijection between the subset $\mathbb{P}(k^3) \setminus \{(0,0,1)\}$ of the projective space $\mathbb{P}(k^3)$ and the set of maximal ideals of $A$.

Proof. The same argument that we showed $m = fA$ in part (a), shows that the map is well-defined. Moreover, part (a) itself shows surjectivity. Suppose that two distinct members $(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$ in $\mathbb{P}(k^3) \setminus \{(0,0,1)\}$ map to the same maximal ideal $m$ in $A$. We may consider the following cases without loss of generality, (Case 1) $\alpha = \alpha' = 1, \beta = \beta', \gamma \neq \gamma'$: This gives a contradiction, since $\gamma - \gamma' \in k^\times \cap m$ cannot hold.

(Case 2) $\alpha = \alpha' = 1, \beta \neq \beta'$: By subtraction, we obtain a real number $y_0$ such that $y_0 \in A/m$. Plugging in $y = y_0$, we obtain a real number $x_0$ such that $x_0 \in A/m$. Furthermore, $x_0^2 + y_0^2 + 1 = 0$ must hold, which is a contradiction.

(Case 3) $\alpha = 1, \beta = 0, \alpha' = 0, \beta' = 1$: This also give two real numbers $x_0, y_0 \in A/m$ with $x_0^2 + y_0^2 + 1 = 0$, which is a contradiction. Hence, the map is injective. \hfill \square

(c) Let $p$ be a non-maximal prime ideal of $A$. Show that the canonical homomorphism $k[X] \rightarrow A$ is finite and injective. Deduce from this that $p \cap k[X] = 0$. Let $g \in p$, and let $g^n + a_{n-1}g^{n-1} + \cdots + a_0 = 0$ be an integral equation with $a_i \in k[X]$. Show that $a_0 = 0$. Conclude that $p = 0$.

Proof. The homomorphism $k[X] \rightarrow A$ is injective, since its kernel is $k[X] \cap (X^2 + Y^2 + 1) = 0$. It is clear that $A$ is an integral extension of $k[X]$. It is also true that $A/p$ is an integral extension of $k[X]/p \cap k[X]$. They are both integral domains. Since $A/p$ is not a field, $k[X]/p \cap k[X]$ is not a field. In $k[X]$, only non-maximal prime ideal is $(0)$. Thus, $p \cap k[X] = 0$.

Suppose $p$ contains a nonzero member $g$. Choose the integral equation for $g$ with minimal degree $n$. From the integral equation, we have $a_0 \in p \cap k[X]$. Thus $a_0 = 0$. Dividing $g$ gives another integral equation for $g$ with degree $n - 1$. This contradicts minimality of $n$. Hence, $p$ must be 0. \hfill \square

1.8 Let $\varphi : A \rightarrow B$ be an integral homomorphism.

(a) Show that $Spec\varphi : SpecB \rightarrow SpecA$ maps a closed point to a closed point, and that any preimage of a closed point is a closed point.

Proof. We use a lemma: $A \subset B$ is integral extension, $A$ and $B$ are integral domains. Then $A$ is a field if and only if $B$ is a field.

Let $p \subset B$ be a prime ideal. Then $\varphi$ induces an embedding $A/\varphi^{-1}(p) \rightarrow B/p$. Since $A/\varphi^{-1}(p)$ and $B/p$ are both integral domains, the lemma applies. Thus, $A/\varphi^{-1}(p)$ is a field if and only if $B/p$ is a field. Equivalently, $\varphi^{-1}(p)$ is a closed point (i.e. maximal ideal) if and only if $p$ is a closed point. \hfill \square

(b) Let $p \in SpecA$. Show that the canonical homomorphism $A_p \rightarrow B \otimes_A A_p$ is integral.
Proof. It is enough to show that any simple tensor $b \otimes_A a$ is integral over $A_p$. Consider an integral equation for $b$, namely $\sum_{i \leq n} a_i b^i = 0$ with $a_i \in A$, $a_0 \neq 0$, and $a_n = 1$. Then, clearly, $(\sum_{i \leq n} a_i b^i) \otimes_A a = 0$. This implies $\sum_{i \leq n} a_i (b \otimes_A a)^i = 0$. Hence $b \otimes_A a$ is integral over $A_p$.

(c) Let $T = \mathcal{V}(A \setminus p)$. Let us suppose that $\mathcal{V}$ is injective. Show that $T$ is a multiplicative subset of $B$, and that $B \otimes_A A_p = T^{-1}B \neq 0$. Deduce from this that $\text{Spec}\varphi$ is surjective if $\mathcal{V}$ is integral and injective.

Proof. Let $\bar{\varphi}$ be the canonical homomorphism in part (b). Note that if $q' \in \text{Max}T^{-1}B$, then $\mathcal{V}^{-1}(q')$ is a maximal ideal in $A_p$. In fact, $\mathcal{V}^{-1}(q') = pA_p$, since $A_p$ is a local ring. Consider the diagram,

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
A_p & \xrightarrow{\bar{\varphi}} & T^{-1}B
\end{array}
\]

Apply $\text{Spec}(-)$ to this diagram,

\[
\begin{array}{ccc}
\text{Spec}T^{-1}B & \xrightarrow{\text{Spec}\bar{\varphi}} & \text{Spec}A_p \\
\downarrow{\text{Spec}\mathcal{V}} & & \downarrow{\text{Spec}\mathcal{V}} \\
\text{Spec}B & \xrightarrow{\text{Spec}\mathcal{V}} & \text{Spec}A
\end{array}
\]

By commutativity, we have

\[\text{Spec}\mathcal{V} \circ \text{Spec}\bar{\varphi}(q') = \text{Spec}\varphi \circ \text{Spec}\mathcal{V}(q') = p.\]

Hence, $\text{Spec}\varphi$ is surjective.

1.9 Let $A$ be a finitely generated algebra over a field $k$.

(a) Let us suppose that $A$ is finite over $k$. Show that $\text{Spec}A$ is a finite set, of cardinality bounded from above by the dimension $\dim_k A$ of $A$ as a vector space. Show that every prime ideal of $A$ is maximal.

Proof. Suppose that we could find $r = \dim_k A + 1$ distinct maximal ideals $m_1, \cdots, m_r$. Then, $m_i + m_j = A$ for $i \neq j$. We can construct a strictly descending chain of ideals $m_1 \supset m_2 \supset \cdots \supset m_1 m_2 \cdots m_r$. This makes a strictly increasing chain of $k$-vector spaces $A/m_1 \subseteq A/m_1 m_2 \subseteq \cdots \subseteq A/m_1 m_2 \cdots m_r$. However, this implies that $r \leq \dim_k A/m_1 m_2 \cdots m_r \leq \dim_k A = r - 1$, which leads to a contradiction. Hence, the number of distinct maximal ideals in $A$ is bounded above by $\dim_k A$.

Now, we show that every prime ideal $p$ in $A$ is indeed maximal. $A/p$ is an integral domain which is finite dimensional $k$-vector space. For any $a \in (A/p) \setminus \{0\}$, the set $\{1, a, a^2, \cdots \}$ is linearly dependent over $k$. Thus, $a$ satisfies a polynomial equation $\sum_{i \leq N} c_i a^i = 0$ with $c_i \in k$ and $c_0 \neq 0$(This is possible, since $A/p$ is an integral domain). This implies that $a \sum_{1 \geq i \geq N} c_i a^{i-1} = -c_0 \in k^\times$. Hence $a$ is invertible in $A/p$, giving that $A/p$ is a field.

(b) Show that $\text{Spec}k[T_1, \cdots, T_d]$ is infinite if $d \geq 1$. 

Proof. We begin with proving that \( \text{Spec} k[T] \) is infinite. If \( k \) is infinite, then \( (T - a) \) is maximal ideal for any \( a \in k \), thus \( \text{Spec} k[T] \) is infinite. If \( k \) is finite, still we have \( \bar{k} \) is infinite. This implies that there are infinitely many irreducible polynomials in \( k[T] \), thus \( \text{Spec} k[T] \) is infinite in this case too.

For the case \( d \geq 2 \), we define an injective function given by

\[
\text{Spec} [T_1] \hookrightarrow \text{Spec} [T_1, \cdots, T_d] \notag
\]

\[
(p(T_1)) \mapsto (p(T_1), T_2, \cdots, T_d) \notag
\]

Hence, \( \text{Spec} [T_1, \cdots, T_d] \) is infinite if \( d \geq 1 \). \( \square \)

(c) Show that \( \text{Spec} A \) is finite if and only if \( A \) is finite over \( k \).

Proof. \( (\Rightarrow) \) This is done in part (a).

\( (\Leftarrow) \) Suppose that \( A \) is infinite over \( k \). Then \( A \) is transcendental over \( k \). Noether Normalization Theorem applies to obtain \( T_1, \cdots, T_d \) which are algebraically independent over \( k \), and an integral extension \( k[T_1, \cdots, T_d] \subseteq k[T_1, \cdots, T_d, t_{d+1}, \cdots, t_r] = A \). Similarly as in (b), we can form an injective map

\[
\text{Spec} [T_1, \cdots, T_d] \rightarrow \text{Spec} [T_1, \cdots, T_d, t_{d+1}, \cdots, t_r] \notag
\]

\[
P \mapsto P + (t_{d+1}, \cdots, t_r). \notag
\]

By part (b), it follows that \( \text{Spec} A \) is infinite. \( \square \)

2.2 Let \( \mathcal{F} \) be a sheaf on \( X \). Let \( s, t \in \mathcal{F}(X) \). Show that the set of \( x \in X \) such that \( s_x = t_x \) is open in \( X \).

Proof. Let \( x \in X \), and \( s_x = t_x \). Then there exists an open neighborhood \( U_x \) of \( x \) such that \( s|_{U_x} = t|_{U_x} \). For any \( y \in U_x \), consider the natural map \( \mathcal{F}(X) \rightarrow \mathcal{F}_y \). Since this natural map is given by \( a \in \mathcal{F}(X) \mapsto [X, a] \) where \([X, a] \) denotes the equivalence class represented by \((X, a)\). But, \( s|_{U_x} = t|_{U_x} \) implies that \( s_y = [X, s] = [U_x, s|_{U_x}] = [U_x, t|_{U_x}] = [X, t] = t_y \). Hence, the set of \( x \in X \) such that \( s_x = t_x \) is open in \( X \). \( \square \)

2.7 Let \( \mathcal{B} \) be a base of open subsets on a topological space \( X \). Let \( \mathcal{F}, \mathcal{G} \) be two sheaves on \( X \). Suppose that for every \( u \in \mathcal{B} \) there exists a homomorphism \( \alpha(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U) \) which is compatible with restrictions. Show that this extends in a unique way to a homomorphism of sheaves \( \alpha : \mathcal{F} \rightarrow \mathcal{G} \). Show that if \( \alpha(U) \) is surjective(resp. injective) for every \( U \in \mathcal{B} \), then \( \alpha \) is surjective (resp. injective).

Proof. Note that every open set \( U \subset X \) is a union of members in \( \mathcal{B} \), say \( U = \cup_i U_i \), where \( U_i \in \mathcal{B} \). Let \( \beta_i \in \mathcal{F}(U_i) \), and \( \beta_j \in \mathcal{F}(U_j) \), and \( \beta_i|_{U_i \cap U_j} = \beta_j|_{U_i \cap U_j} \) for all \( i, j \), which data uniquely determine \( \beta \in \mathcal{F}(U) \) (\( \therefore \) \( \mathcal{F} \) is a sheaf). We know that \( g_i = \alpha(U_i)(\beta_i) \in \mathcal{G}(U_i) \), \( g_j = \alpha(U_j)(\beta_j) \in \mathcal{G}(U_j) \), and \( \alpha(U_i \cap U_j)(\beta_i|_{U_i \cap U_j}) = \alpha(U_i \cap U_j)(\beta_j|_{U_i \cap U_j}) \in \mathcal{G}(U_i \cap U_j) \). Since the homomorphism \( \alpha \) is compatible with restrictions on members of \( \mathcal{B} \), we have

\[
g_i|_{U_i \cap U_j} = \alpha(U_i \cap U_j)(\beta_i|_{U_i \cap U_j}) = \alpha(U_i \cap U_j)(\beta_j|_{U_i \cap U_j}) = g_j|_{U_i \cap U_j}. \notag
\]

Thus, this data uniquely determine \( g \in \mathcal{G}(U) \) (\( \therefore \) \( \mathcal{G} \) is a sheaf). Define \( \alpha(U)(\beta) = g \), then this is the unique extension to a homomorphism of sheaves \( \alpha : \mathcal{F} \rightarrow \mathcal{G} \).

We use the fact that sequence of sheaves \( \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \) is exact if and only if \( \mathcal{F}_x \rightarrow \mathcal{G}_x \rightarrow \mathcal{H}_x \) is exact for each \( x \in X \). In fact, \( \alpha(U) \) is surjective (resp. injective) for every \( U \in \mathcal{B} \) imply \( \mathcal{F}_x \rightarrow \mathcal{G}_x \rightarrow 0 \) (resp. \( 0 \rightarrow \mathcal{F}_x \rightarrow \mathcal{G}_x \)) is exact.
Proof. We define such that glueing data \( i,j \) \{ an exact sequence in sheaves. Hence \( \text{Equivalence of (1),(2), and (3) is direct from definition. We show the equivalence } \)
\( P \)
\( \text{has no square factor.} \)
\( \text{Moreover, continuous function with this property has to be unique.} \)
\( \text{P} \)
\( \text{reduced (resp. irreducible; resp. integral) if and only if} \)
\( \text{f} \)
\( A \)
\( O \)
\( \text{lent:} \)
\( \text{Proof. We define} \ f : X \rightarrow Y \text{ by} \ f(x) = f_i(x) \text{ where} \ x \in U_i \text{ for some } i. \text{ The glueing data} \ f_1|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ gives that} \ f : X \rightarrow Y \text{ is a continuous function. Moreover, continuous function with this property has to be unique.} \)
\( \text{The morphism of sheaves} \ f_*^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_{U_i} \text{ gives the ring homomorphism} \)
\( f_*^\#(V) : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_{U_i}(f_*^{-1}(V)) = \mathcal{O}_X(f^{-1}(V) \cap U_i). \text{ The glueing data gives that} \)
\( f_2|_{U_i \cap U_j}^\#(V) : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V) \cap U_i \cap U_j) = f_1|_{U_i \cap U_j}^\#(V) : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V) \cap U_i \cap U_j). \)
\( \text{Together with the compatibility of restriction,} \ a \in \mathcal{O}_Y(V) \rightarrow a_i \in \mathcal{O}_X(f^{-1}(V) \cap U_i) \text{ has the glueing data} \ a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}. \text{ Since} \)\( \mathcal{O}_X(f^{-1}(V) \cap -) \) \text{ is a sheaf, we obtain a unique} \ a \in \mathcal{O}_X(f^{-1}(V)) \text{ such that} \ a|_{U_i} = a_i \text{ for each} \ i. \text{ Define} \ f_*^\# : \mathcal{O}_Y \rightarrow \mathcal{O}_X \text{ by} \ a \in \mathcal{O}_Y(V) \rightarrow a \in \mathcal{O}_X(f^{-1}(V)). \text{ Then} \ (f_*^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \text{ is the desired morphism of ringed topological spaces.} \)
\( \text{3.19 Let} \ K \text{ be a number field. Let} \ \mathcal{O}_K \text{ be the ring of integers of} \ K. \text{ Using the finiteness theorem of the class group} \ cl(K), \text{ show that every open subset of} \ Spec \mathcal{O}_K \text{ is principal. Deduce from this that} \)\( \text{every open subscheme of} \ Spec \mathcal{O}_K \text{ is affine.} \)
\( \text{Proof. Let} \ U \text{ be an open subset of} \ Spec \mathcal{O}_K, \text{ then} (Spec \mathcal{O}_K) \backslash U = V(I) \text{ for some ideal} I \subset \mathcal{O}_K. \text{ This ideal} I \text{ belongs to an ideal class} [Q] \subset cl(K). \text{ Thus, there are} \)
\( a, b \in \mathcal{O}_K \backslash \{0\} \) \text{ such that} \( aI = bQ. \text{ This implies that} I = (b/a)Q \text{ and there is some} \ n \geq 0 \text{ such that} \ I^n = (b/a)^n Q^n = \gamma \mathcal{O}_K \text{ for some} \ \gamma \in K \text{ (In particular, we can take} \ n = \#cl(K). \text{ Since} \ I^n \subset \mathcal{O}_K, \text{ we have} \ \gamma \mathcal{O}_K \subset \mathcal{O}_K. \text{ This proves that} \ \gamma \in \mathcal{O}_K,\text{ and} \ I^n \text{ is a principal ideal. Hence,} \ V(I) = V(I^n) = V(\gamma \mathcal{O}_K) \text{ is a principal closed subset, or equivalently,} U = D(\gamma) \text{ is a principal open set.} \)
\( \text{Therefore, every open subscheme} U = D(\gamma) \text{ of} \ Spec \mathcal{O}_K \text{ is isomorphic to} Spec \mathcal{O}_K[1/\gamma], \text{ which is affine.} \)
\( \text{4.1 Let} \ k \text{ be a field and} P \in k[T_1, \cdots, T_n]. \text{ Show that} Spec(k[T_1, \cdots, T_n]/(P)) \text{ is reduced (resp. irreducible; resp. integral) if and only if} P \text{ has no square factor (resp. admits only one irreducible factor; resp. is irreducible).} \)
\( \text{Proof. Let} A = k[T_1, \cdots, T_n]/(P), \text{ and} q \in Spec A. \text{ Then the following are equivalent:} \)
\( 1) \mathcal{O}_{A,q} = A_q \text{ is reduced for all} q \in Spec A. \)
\( 2) \text{Only nilpotent element in} A_q \text{ is 0.} \)
\( 3) \cap \{ q' \in Spec k[T_1, \cdots, T_n] : (P) \subset q' \subset q \} = (P). \)
\( 4) P \text{ has no square factor.} \)
\( \text{Equivalence of (1),(2), and (3) is direct from definition. We show the equivalence of (3), and (4). This follows from} \)
\( \cap \{ q' \in Spec k[T_1, \cdots, T_n] : (P) \subset q' \subset q \} = (P_0), \text{ where} P_0 \text{ is the product of all irreducible factors of} P. \)
Remark that Spec\(A \neq \emptyset\) is irreducible if and only if \(\sqrt{(0)}\) is prime. Equivalently,
\[
\cap \{ q' \in \text{Speck}[T_1, \cdots, T_n] : (P) \subset q' \} = (P_0)
\]
is prime. Similarly as before, \(P_0\) is the product of all irreducible factors of \(P\). Now, \((P_0)\) is prime if and only if there is only one irreducible factor in \(P\).

\(A\) is integral if and only if \(A\) is both reduced and irreducible. Therefore, \(A\) is integral if and only if \(P\) has no square factor and admits only one irreducible factor, i.e. \(P\) is irreducible.

\(\square\)