Problem 1.  
\(\Leftarrow\) We do this in two steps: 
(1) If \(A\) has a composition series, then \(A\) is noetherian and artinian. 
(2) \(A\) has an \(A\)-composition series. 

Proof of (1). 
Suppose \(A\) has a composition series \(S\) of length \(n\). If either chain condition fails to hold, one can find submodules 
\[A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1},\]
which form a normal series \(T\) of length \(n + 1\). By Shreier’s theorem, \(S\) and \(T\) have refinements that are equivalent. This is a contradiction since \(S\) has length \(n\), and refinement of \(T\) has length at least \(n + 1\). Therefore, \(A\) satisfies both chain conditions. 

Proof of (2).  
Using the fact that \(A\) is noetherian, any ideal of \(A\) contains a finite product of prime ideals. In particular, the ideal \((0)\) contains a finite product \(m_1 \cdots m_n\) of prime ideals \(m_1, \cdots, m_n\) which are also maximal ideals. Thus, we have a filtration 
\[0 = m_1 \cdots m_n \subset m_1 \cdots m_{n-1} \subset \cdots \subset m_1 \subset A.\]
Each quotient \(m_1 \cdots m_i / m_1 \cdots m_{i+1}\) is a finitely generated module over a field \(A/m_{i+1}\), and therefore has an \(A\)-composition series. It follows that \(A\)-module \(A\) has an \(A\)-composition series, so \(A\) is artinian by (1). 

\(\Rightarrow\) We do this in two steps: 
(3) Left artinian DOES imply left noetherian for noncommutative rings too. 
(4) If \(A\) is an artinian commutative ring, then \(\text{Spec}(A) = \text{Max}(A)\). 

Proof of (3). 
Note that if \(A\) is artinian, then \(\tilde{A} = A/\text{Rad}(A)\) is semisimple, and \(\text{Rad}(A)\) is nilpotent.(cf. Homework6 Problem4.) For \(J = \text{Rad}(A)\), fix \(n\) such that \(J^n = 0\). Consider the filtration 
\[A \supset J \supset J^2 \supset \cdots \supset J^n = 0.\]
It is enough to show that \(J^i / J^{i+1}\) has a composition series. But \(J^i / J^{i+1}\) is artinian as module over \(\tilde{A}\). Since \(\tilde{A}\) is semisimple, \(J^i / J^{i+1}\) is semisimple \(\tilde{A}\)-module, so it is a direct sum of simple \(\tilde{A}\)-modules. The chain condition on \(J^i / J^{i+1}\) implies that this direct sum must be finite, so \(J^i / J^{i+1}\) does have a composition series as \(A\)-module. Hence, \(A\) has a composition series as \(A\)-module. By (1), it follows that \(A\) is left noetherian. 

Proof of (4). 
Let \(p \subset A\) be a prime ideal of \(A\). Then \(A/p\) is an integral domain. Thus, it is
enough to show that artinian integral domain is a field. Let \( a \in A/\mathfrak{p} \) be a nonzero element. Consider a chain of ideals in \( A/\mathfrak{p} \),

\[
(a) \supset (a^2) \supset \cdots
\]

Since \( A/\mathfrak{p} \) is artinian, this chain has to stop, say \( (a^n) = (a^{n+1}) \). Then \( a^n = ba^{n+1} \) for some \( b \in A/\mathfrak{p} \). This implies that \( a^n(1-ba) = 0 \). Since \( a^n \neq 0 \), we must have \( 1-ba = 0 \). Hence \( A/\mathfrak{p} \) is a field.

**Problem 2.**

A fractional ideal \( I \) is a nonzero \( A \)-submodule \( I \) such that \( aI \subset A \) for some nonzero \( a \in A \). Let \( J \) be a fractional ideal in \( A \). Denote

\[
J^{-1} = \{ a \in K | aJ \subset A \}.
\]

Let \( P \) be the unique nonzero prime ideal in \( A \). Since \( A \) is local, we have \( P = \mathfrak{m} \). It is enough to show that \( A \) is a PID. This require the following facts.

1. Let \( K \) be the quotient field of \( A \), for a fractional ideal \( I \) in \( A \), then

\[
\bar{I} = \{ a \in K | aI \subset I \} = A.
\]

2. \( A \nsubseteq P^{-1} \).
3. \( P \) is invertible.
4. \( \cap_{n \geq 0} P^n = 0 \).
5. \( P \) is principal.

We assume (1), \( \cdots \), (5), let \( I \) be a proper ideal in \( A \). Then \( I \subset P \). By (4), we can find \( N \) such that \( I \subset P^N \), and \( I \nsubseteq P^{N+1} \). By (5), there is \( a \in A \) such that \( P = (a) \).

We see that \( I \subset P^N = (a^N) \). Choose \( b \in I - P^{N+1} \). Since \( b \in P^N \), we can find \( u \in A \) such that \( b = ua^N \). It follows that \( u \) has to be a unit in \( A \), otherwise we would have \( u \in P \), and \( b \in P^{N+1} \). Thus, \( (a^N) = (ua^N) = (b) \subset I \), and we obtain \( I = (a^N) \).

**Proof of (1).**

The inclusion \( \supseteq \) is obvious. Since \( \bar{I} \) is a fractional ideal, it is isomorphic to some ideal in \( A \). So, \( \bar{I} \) is finitely generated \( A \)-module, since \( A \) is noetherian. This means that every \( x \in \bar{I} \) is integral over \( A \). Since \( A \) is integrally closed, \( x \in A \). Thus, we obtain that \( \bar{I} = A \).

**Proof of (2).**

For any ideal \( J \subset A \), we have \( A \subset J^{-1} \). Let \( \mathcal{F} \) be a family of all ideals \( J \) such that \( A \subsetneq J^{-1} \). Choose a nonzero \( a \in P \), then we know that \( a \) is nonunit, and let \( J = (a) \). Then \( 1/a \in J^{-1} \), but \( 1/a \notin A \). So, the family \( \mathcal{F} \) is nonempty. Since \( A \) is noetherian, we can find a maximal element \( M \) in the family \( \mathcal{F} \). We claim that \( M \) is a prime ideal. For, assume that \( ab \in M \) but \( a \notin M \). Choose \( c \in M^{-1} - A \), then \( cab \in A \), hence \( bc(aA + M) \subset A \), giving that \( bc \in (aA + M)^{-1} \). By maximality of \( M \), we have \( bc \in A \). Thus, \( c(bA + M) \subset A \), which implies \( c \in (bA + M)^{-1} \). Since \( c \notin A \), and again by maximality of \( M \), we have \( bA + M = M \). Hence \( b \in M \), and \( M \) is a prime ideal. By uniqueness, we must have \( M = P \). Therefore, \( A \subsetneq P^{-1} \).

**Proof of (3).**

Clearly \( P \subset PP^{-1} \subset A \). Since \( A \) is local, \( PP^{-1} = P \) or \( A \). But if \( PP^{-1} = P \) then \( A \subsetneq P^{-1} \subset \bar{P} = A \), by (1) and (2). This is a contradiction. Hence \( PP^{-1} = A \).

**Proof of (4).**
This is a consequence of Artin-Rees Lemma, and Nakayama Lemma. (cf. Homework3 Problem1).

Proof of (5).
Choose $a \in P - P^2$ (possible by (4)). Then $aP^{-1}$ is a nonzero ideal in $A$ such that $aP^{-1} \not\subseteq P$. Since every proper ideal of $A$ is contained in $P$, we must have $aP^{-1} = A$. Hence, $(a) = aA = aP^{-1}P = AP = P$ by (3).

Problem 3.
$(\text{char} K = 0)$ We start with a fixed subfield $F$ of the complex field $K$. Let $\mu$ be an irreducible character afforded by a simple $KG$-module $M$. Let $F(\mu)$ be the field obtained by adjoining all $\mu(g)$ for $g \in G$. Consider those fields $S$ such that $F \subset S \subset K$ for which there exists a $SG$-module $V$ such that $M \cong K \otimes_S V$.
For such field $S$, any $S$-basis for $V$ becomes a $K$-basis for $M$. Thus, we see that $\mu(g) \in S$ for all $g \in G$, since each $\mu(g)$ is trace of a matrix in $S$. Thus, we must have $F(\mu) \subset S \subset K$.
The Schur index is defined by
$m_F(\mu) = \min(S : F(\mu))$
where the minimum is taken over all fields $S$ such that $M$ is realizable in $S$. Then $m_F(\mu) = m_{F(\mu)}(\mu)$. Further, $m_F(\mu) = 1$ if and only if $M$ affords a representation realizable in $F(\mu)$.

Let $M$ be an irreducible $KG$-module. Let $S$ be any subfield of $K$. Then $M$ is a direct summand of $V^K = K \otimes_S V$. Thus, every irreducible $K$-representation of $G$ is realizable in $S$ if and only if $S$ is a splitting field of $G$. The following holds for Schur index;
Let $M$ be an irreducible $KG$-module with character $\mu$, and $F$ be a subfield of $K$. Then for each $FG$-module $W$, the multiplicity with which $M$ occurs as a factor of $W^K$ is a multiple of $m_F(\mu)$.

Let $F = \mathbb{Q}(\sqrt[n]{1})$ where $n$ is the exponent of $G$. Then $\mu(g) \in F$ for all $g \in G$, so $F(\mu) = F$. We claim that $F$ is a splitting field of $G$. We need only to show that $m_F(\mu) = 1$ for each $\mu$. By Brauer’s theorem on induced characters, we may write
$\mu = \sum a_i w_i^G$
where $a_i \in \mathbb{Z}$, and $w_i$ are one dimensional characters of elementary subgroups of $G$. (Elementary subgroup is a product of cyclic group and $p$-group) Now every one dimensional representation of a subgroup of $G$ is realizable in $F$. Thus each $w_i^G$ is a character some $FG$-module, and so the multiplicity of $\mu$ in each $w_i^G$ is a multiple of $m_F(\mu)$. However, the multiplicity of $\mu$ in each $w_i^G$ is 1. Therefore $m_F(\mu) = 1$.
If $K$ contains a primitive $n$-th root of unity, then by tensoring with $K$, we obtain that $K$ is a splitting field of $G$.

Problem 4.
The character table for $G$ has to contain the following two more rows
Since the character table contains a 2-dimensional irreducible representation, it has to contain two more rows (by multiplying with 1-dimensional irreducible representations)

<table>
<thead>
<tr>
<th>$G$</th>
<th>${1}$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1^2$</td>
<td>1 1 1 1 1 1 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$w$ $w$ $w$ $w$ $w$ $w$ $w$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, we have the following table so far, (0 on the last row comes from orthogonality with the first column.)

<table>
<thead>
<tr>
<th>$G$</th>
<th>${1}$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1^2$</td>
<td>1 1 1 1 1 1 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$w$ $w$ $w$ $w$ $w$ $w$ $w$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let $a = |C_4|, b = |C_5|, c = |C_6|, and d = |C_7|$. Then by orthogonality relations of the first row and the second, third, we have

\[ wa + w^2b + wc + w^2d = w^2a + wb + w^2c + wd. \]

This implies $a + c = b + d$. We use orthogonality relations of the first row and the fourth, fifth, we have

\[ -a - b + c + d = -w^2a - wb + w^2c + wd. \]

This implies $(-a + c)(1 - w^2) = (-b + d)(w - 1)$. This forces $a = c, b = d,$ and together with the above equation, we have $a = b = c = d$. The orthogonality relation of the first column and the third column, we obtain $e | 3$. Thus $e = 1$ or $e = 3$. But, $e = 1$ gives $|G| = 16$. Since the character table contains $w, G$ must contain some element having order $k$ with $3|k$. Therefore, we must have $e = 3$. Then by orthogonality of the first column and the second column, we have $f = 3, g = -1$. Now, we have that $|G| = 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + 3^2 = 24$. The orthogonality relation of the first and fourth rows, we obtain $|C_2| = 1$. The orthogonality of the first and last rows give $|C_3| = 6$. Since we have $a = b = c = d$, comparing with the order of $G$ gives $a = b = c = d = 4$. Hence, the character table is

<table>
<thead>
<tr>
<th>$G$</th>
<th>${1}$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
<th>$C_6$</th>
<th>$C_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1^2$</td>
<td>1 1 1 1 1 1 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$w$ $w$ $w$ $w$ $w$ $w$ $w$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The information on $G$ is so far,

(a) $|G| = 24$. 
(b) \( G \) has 7 conjugacy classes.
(c) The sizes of conjugacy classes are 1,1,6,4,4,4,4.

To determine the group \( G \), we use the list of all groups of order 24.

1. \( \mathbb{Z}/24\mathbb{Z} \)
2. \( \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)
3. \( \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)
4. \( S_4 \)
5. \( SL_2(\mathbb{Z}/3\mathbb{Z}) \)
6. \( D_{24} \)
7. \( \mathbb{Z}/2\mathbb{Z} \times A_4 \)
8. \( \mathbb{Z}/2\mathbb{Z} \times D_{12} \)
9. \( \mathbb{Z}/2\mathbb{Z} \times T \) (\( T \) is tetrahedron symmetry.)
10. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times D_8 \)
11. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times S_3 \)
12. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times S_3 \)
13. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times T \) (\( T \) is tetrahedron symmetry.)
14. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\math2

From (a),(b) above, we know that \( G \) is not abelian. So, we exclude (1),(2),(3). Since \( S_4 \) has 5 conjugacy classes, we exclude (4). Since \( D_{24} \) has a conjugacy class having size 2, we exclude (6). The number of conjugacy classes is 7 which is a prime number. Thus, we must exclude all groups which are direct product of cyclic group and another group. So, (7),\( \cdots \), (12) are excluded. Using the multiplicative structure of semidirect product, we obtain that (13),(14),(15) have a conjugacy class of size 2. The only thing remains is (5), \( SL_2(\mathbb{Z}/3\mathbb{Z}) \).

We have a general result about the conjugacy classes of \( SL_2(\mathbb{F}_q) \), where \( q \) is odd.

<table>
<thead>
<tr>
<th>Representative</th>
<th>no.of classes</th>
<th>size of class</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm 1 \begin{pmatrix} 0 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \pm 1 \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>2</td>
<td>( (q-1)(q+1) )</td>
</tr>
<tr>
<td>( \pm 1 \begin{pmatrix} \epsilon &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>2</td>
<td>( (q-1)(q+1) )</td>
</tr>
<tr>
<td>( a \begin{pmatrix} 0 &amp; a^{-1} \ 0 &amp; 1 \end{pmatrix} ) ( a \in \mathbb{F}_q^* \setminus {\pm 1} )</td>
<td>( q-3 )</td>
<td>( q(q+1) )</td>
</tr>
<tr>
<td>( 0 \begin{pmatrix} -1 &amp; 1 \ 1 &amp; \alpha \end{pmatrix} )</td>
<td>( q-1 )</td>
<td>( q(q-1) )</td>
</tr>
</tbody>
</table>

\( p(t) = t^2 - \alpha t + 1 \in \mathbb{F}_q[t] \) is irreducible.

From this, we can verify that \( SL_2(\mathbb{Z}/3\mathbb{Z}) \) has 7 conjugacy classes having sizes 1,1,6,4,4,4,4. Further, there is only one way to express 24 as the sum of 1 and 6 other squares, namely \( 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + 3^2 \). The Artin-Wedderburn decomposition for \( C SL_2(\mathbb{Z}/3\mathbb{Z}) \) is

\[
C SL_2(\mathbb{Z}/3\mathbb{Z}) = C \times C \times C \times M_2(\mathbb{C}) \times M_2(\mathbb{C}) \times M_2(\mathbb{C}) \times M_3(\mathbb{C}).
\]
Hence the group $SL_2(\mathbb{Z}/3\mathbb{Z})$ must have the same 7 by 7 character table as above.

**Problem 5.**

- **Solution of Cubic Equations by Radicals:**
  Consider a cubic polynomial $f(x) = x^3 + ax^2 + bx + c$. By substitution $x = y - a/3$, we have $f(x) = g(y) = y^3 + py + q$, where
  $$p = \frac{1}{3}(3b-a^2), \quad q = \frac{1}{27}(2a^3 - 9ab + 27c).$$

  Let $\alpha, \beta, \gamma$ be all roots of $g(y) = 0$. We want to express roots of $g(y) = 0$ in the form $A + B$ where $-3AB = p$. Using $A^3 + B^3 = (A + B)^3 - 3AB(A + B)$, we obtain
  $$A^3 + B^3 = -q.$$ 

  Also, we have $A^3B^3 = -\frac{p^3}{27}$. Thus, $A^3$, and $B^3$ are roots of the quadratic
  $$t^2 + qt - \frac{p^3}{27} = 0.$$ 

  From the formula for solution of quadratic equations,
  $$A^3 = \frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2},$$ 
  $$B^3 = \frac{-q - \sqrt{q^2 + \frac{4}{27}p^3}}{2}.$$ 

  Choosing appropriate cubic roots of RHS which has to satisfy $-3AB = p$, we obtain that
  $$\alpha = A + B, \quad \beta = wA + w^2B, \quad \gamma = w^2A + wB.$$ 

  where $w = \exp(2\pi i/3)$.

- **Solution of Quartic Equations by Radicals:**
  Consider a quartic polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$. By substitution $x = y - a/4$, we have $f(x) = g(y) = y^4 + py^2 + qy + r$, where
  $$p = \frac{1}{8}(-3a^2 + 8b),$$ 
  $$q = \frac{1}{8}(a^3 - 4ab + 8c),$$ 
  $$r = \frac{1}{256}(-3a^4 + 16a^2b - 64ac + 256d).$$

  Let $\alpha, \beta, \gamma, \delta$ be all roots of $g(y) = 0$. Add $(ly + m)^2$ to the equation $g(y) = 0$, then we obtain
  $$y^4 + py^2 + qy + r + (ly + m)^2 = (ly + m)^2.$$ 

  We use $(y^2 + \lambda)^2 = y^4 + 2\lambda y^2 + \lambda^2$, to complete the square of LHS above, which is
  $$y^4 + (p + l^2)y^2 + (q + 2lm)y + r + m^2 = (ly + m)^2.$$ 

  Comparing the coefficient,
  $$2\lambda = p + l^2, \quad q + 2lm = 0, \quad r + m^2 = \lambda^2.$$ 

  Then, it follows that
  $$(2\lambda - p)(\lambda^2 - r) = \frac{q^2}{4}.$$
Since this is a cubic equation for \( \lambda \), let \( \lambda \) be a real root for this cubic equation, which can be solved by radicals. Then we have

\[(y^2 + \lambda)^2 = (ly + m)^2\]

where \( l = \sqrt{2\lambda - p} \), and \( m = \sqrt{\lambda^2 - r} \). Now, we reduced the quartic to two quadratics, \( y^2 + \lambda = ly + m \), and \( y^2 + \lambda = -ly - m \). Hence, the general solution for \( g(y) = 0 \) is

\[
\alpha = \frac{l + \sqrt{l^2 - 4(\lambda - m)}}{2}, \quad \beta = \frac{l - \sqrt{l^2 - 4(\lambda - m)}}{2},
\]

\[
\gamma = \frac{-l + \sqrt{l^2 - 4(\lambda + m)}}{2}, \quad \delta = \frac{-l - \sqrt{l^2 - 4(\lambda + m)}}{2}.
\]