Please provide complete and well-written solutions to the following exercises.

Due May 28, in the discussion section.

Assignment 7

**Exercise 1.** Let $A$ be an $m \times n$ matrix. Let $B$ be an $\ell \times m$ matrix. Show that $(BA)^t = A^tB^t$.

**Exercise 2.** Let $n \in \mathbb{N}$. Let $S_n$ denote the set of permutations on $\{1, \ldots, n\}$. For any $\sigma \in S_n$, define $\text{sign}(\sigma) := (-1)^N$, where $\sigma$ can be written as the product of $N$ transpositions.

Now, let $A$ be an $n \times n$ matrix with entries $A_{ij}, i, j \in \{1, \ldots, n\}$. Consider the expression

$$F(A) := \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} A_{i \sigma(i)}.$$

(a) Let $A$ be a $2 \times 2$ matrix. Show directly that $F(A) = \det(A)$. (Hint: there are only two elements in $S_2$. What are they?)

(b) Show that for any $n \times n$ matrix, $F(A) = \det(A)$.

**Exercise 3.** Let $A$ denote the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 5 & 0 \end{pmatrix}.$$

Compute $\det(A)$. Explain what formula you are using, and why your computation of $\det(A)$ is correct. (Hint: use the previous exercise.)

**Exercise 4.** Let $\mathbf{F}$ be a field, and let $M$ be an $n \times n$ matrix with entries in the field $\mathbf{F}$. Suppose there exist matrices $A, B$ such that $A$ is a square matrix, and such that $M$ can be written as

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}.$$

Show that $\det(M) = \det(A)$.

**Exercise 5.** Let $\mathbf{F}$ be a field, and let $M$ be an $n \times n$ matrix with entries in the field $\mathbf{F}$. Suppose there exist matrices $A, B, C$ such that $A$ is a square matrix, and such that $M$ can be written as

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Show that $\det(M) = \det(A) \cdot \det(C)$. (Hint: consider the matrix product $\begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$, and apply the previous exercise to the result.)
Exercise 6. Define
\[ A := \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}. \]

- Find all of the eigenvalues of \( A \).
- For each eigenvalue \( \lambda \) of \( A \), find the set of eigenvectors corresponding to \( \lambda \).
- Find a basis for \( \mathbb{R}^2 \) consisting of eigenvectors of \( A \) (if possible).
- If you can find a basis of \( \mathbb{R}^2 \) consisting of eigenvectors of \( A \), then find an invertible matrix \( Q \) and a diagonal matrix \( D \) such that \( Q^{-1}AQ = D \).

Exercise 7. Section 5.1, Exercise 8 in the textbook. (The phrase “\( T \) is a linear operator on a vector space \( V \)” means that \( T : V \to V \) is a linear transformation.)

Exercise 8. Let \( T \) be a linear transformation on a vector space \( V \), and let \( x \) be an eigenvector of \( T \) corresponding to the eigenvalue \( \lambda \). For any positive integer \( m \), prove that \( x \) is an eigenvector of \( T^m \) corresponding to the eigenvalue \( \lambda^m \). Then, state and prove an analogous result for matrices.

Exercise 9. Define \( T : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R}) \) by \( T(A) := A^t \). Note that \( T \) is a linear transformation.

- Show that \( \pm 1 \) are the only eigenvalues of \( T \).
- Describe the eigenvectors corresponding to each eigenvalue of \( T \).
- Find an ordered basis \( \beta \) for \( M_{2 \times 2}(\mathbb{R}) \) such that \( [T]_\beta^\beta \) is a diagonal matrix.
- Find an ordered basis \( \beta \) for \( M_{n \times n}(\mathbb{R}) \) such that \( [T]_\beta^\beta \) is a diagonal matrix for \( n > 2 \).

Exercise 10. Section 5.2, Exercise 2(ab) in the textbook.

Exercise 11. Consider the following 2 \( \times \) 2 matrix.
\[ A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}. \]

Let \( n \) be an arbitrary positive integer. Find an expression for \( A^n \).