Please provide complete and well-written solutions to the following exercises.
Due April 9, in the discussion section.

Assignment 1

Exercise 1. Section 1.2, Exercise 1(abghk) in the textbook.

Exercise 2. Let \( V \) be a vector space over a field \( F \). Using the axiomatic definitions of fields and vector spaces, prove the following facts:

- \( \forall \ v \in V, \ 0 \cdot v = 0. \)
- \( \forall \ v \in V, \ (-1) \cdot v = -v. \)
- \( \forall \ \alpha \in F, \ \text{and for } 0 \in V, \ \alpha \cdot 0 = 0. \)
- \( \forall \ \alpha \in F, \ \forall \ v \in V, \ \alpha \cdot (-v) = (-\alpha) \cdot v = -(\alpha \cdot v). \)

Exercise 3. Section 1.3, Exercise 8(abf) in the textbook.

Exercise 4. Show that the intersection of two subspace is also a subspace.

Exercise 5. Section 1.4, Exercise 3(c) in the textbook.

Exercise 6. Section 1.5, Exercise 1(abdef) in the textbook.

Exercise 7. Let \( V \) be a vector space over a field \( F \). Let \( \{u_1, \ldots, u_n\} \subseteq V \) satisfy the following property. For any \( u \in V \), there exist unique scalars \( \alpha_1, \ldots, \alpha_n \in F \) such that \( u = \alpha_1 u_1 + \cdots + \alpha_n u_n. \)

Prove that \( \{u_1, \ldots, u_n\} \) is a basis of \( V \).

Exercise 8. Give an example of a subset of \( \mathbb{R}^2 \) that is closed under vector addition, but which is not closed under multiplication by scalars.

Exercise 9. Give an example of a subset of \( \mathbb{R}^2 \) that is closed under scalar multiplication, but which is not closed under vector addition.

Exercise 10. Find three nonzero, distinct vectors \( f, g, h \in \mathbb{R}^3 \) satisfying the following properties: \( \text{span}(f, g) = \text{span}(g, h) = \text{span}(f, g, h) \), and \( \text{span}(f, h) \neq \text{span}(f, g, h) \).

Exercise 11. Consider the subset of the integers \( X = \{0, 1, 2, \ldots, 18, 19\} \). For any \( x, y \in X \), define the addition operation \( x + y := (x + y) \mod 20 \), where \( (x + y) \) denotes addition in \( \mathbb{Z} \). For any \( x, y \in X \), define the multiplication operation \( x \cdot y := (xy) \mod 20 \), where \( (xy) \) denotes multiplication in \( \mathbb{Z} \). Note that \( X \) is now closed under multiplication and addition with these definitions. (For example, \( 14 + 12 = 6 \mod 20, 13 \cdot 7 = 11 \mod 20 \).) Is \( X \) a field? Prove your assertion.

Exercise 12. Consider the set \( F = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \). Prove that \( F \) is a field.