This homework should be submitted just before the beginning of class, on March 26th, 2012. You should bring a copy of your homework to class, in order to participate in class discussion around your homework.

Please read carefully the following instructions:

This homework is based on questions from the midterm exam and on your responses to them. For three problems (1, 6, 7) we offer a skeleton of a proof, and you are required to add all the missing parts, including the Given, the RTP, and the justification for each step.

For two problems (2, 3) we offer a full proof. For these problems we provide a sample of responses that have flaws, inaccuracies, and/or redundancy. You need to point to all the flaws, inaccuracies and redundant (unnecessary) steps in the responses and explain why you regard them as such.
1. Let $n \in \mathbb{Z}$. Prove that if $5n - 7$ is even then $n$ is odd.

Given:

RTP:

A skeleton of a proof:

$$5n - 7 = 2k$$

$$5n = 2k + 7$$

odd $\times$ n = odd

n = odd
6. Let $x, y \in \mathbb{Z}$. 

(a) Prove that $(x^2 - y^2)$ is divisible by 4 if $x$ and $y$ are of the same parity (i.e., either both $x$ and $y$ are even or both $x$ and $y$ are odd).

Given:

RTP:

A skeleton of a proof:

\[ x + y = 2n \]
\[ x - y = 2k \]
\[ x^2 - y^2 = 4nk \]
\[ \frac{x^2 - y^2}{4} = nk \]
7. (a) Prove that for any two positive numbers $x$, $y$, their arithmetic mean is larger than or equal to their geometric mean, i.e.: $\sqrt{x \cdot y} \leq \frac{x + y}{2}$

Given:

RTP:

A skeleton of a proof:

$$(x - y)^2 \geq 0$$

$$x^2 + y^2 - 2xy \geq 0$$

$$x^2 + y^2 + 2xy \geq 4xy$$

$$(x + y)^2 \geq 4xy$$

$$\frac{x+y}{2} \geq \sqrt{xy}$$

(b) When are these two means equal? That is, under what conditions does $\sqrt{x \cdot y} = \frac{x+y}{2}$ for positive numbers $x$ and $y$? Prove your claim.

Given:

RTP:

A skeleton of a proof:

$$\frac{x+y}{2} = \sqrt{xy}$$

$$x + y = 2\sqrt{xy}$$

$$x^2 + y^2 + 2xy = 4xy$$

$$(x - y)^2 = 0$$

$$x = y$$
2. Let \( x, y \in R \). Prove that \(|x \cdot y| = |x| \cdot |y|\).

**Given:** \( x, y \in R \)

**RTP:** \(|x \cdot y| = |x| \cdot |y|\)

**A Proof:** We use the following definition of the Absolute Value of a real number \( r \):

\[
|r| = \begin{cases} 
  r, & \text{if } r \geq 0 \\
  -r, & \text{if } r \leq 0
\end{cases}
\]

*There are 3 cases we need to check:*

**Case 1:** \( x \geq 0, y \geq 0 \);

**Case 2:** \( x \geq 0, y \leq 0 \);

**Case 3:** \( x \leq 0, y \leq 0 \)

- **Case 1:**
  \[
x \geq 0 \Rightarrow |x| = x \]
  \[
y \geq 0 \Rightarrow |y| = y
  \Rightarrow |x| \cdot |y| = x \cdot y
  
  \[
x \geq 0, y \geq 0 \Rightarrow x \cdot y \geq 0 \Rightarrow |x \cdot y| = x \cdot y
  
  \[
\]

- **Case 2:**
  \[
x \geq 0 \Rightarrow |x| = x
  \]
  \[
y \leq 0 \Rightarrow |y| = -y
  \Rightarrow |x| \cdot |y| = x \cdot (-y) = -(x \cdot y)
  
  \[
x \geq 0, y \leq 0 \Rightarrow x \cdot y \leq 0 \Rightarrow |x \cdot y| = -(x \cdot y)
  
  \[
\]

- **Case 3:**
  \[
x \leq 0 \Rightarrow |x| = -x
  \]
  \[
y \leq 0 \Rightarrow |y| = -y
  \Rightarrow |x| \cdot |y| = (-x) \cdot (-y) = x \cdot y
  
  \[
x \leq 0, y \leq 0 \Rightarrow x \cdot y \leq 0 \Rightarrow |x \cdot y| = x \cdot y
  
  \[
\]

**Q.E.D.**
SAMPLE RESPONSES FOR PROBLEM 2:

Response 2.1:

Case 1
Let \( x = -a \) and \( y = -b \)

\[
|a \cdot -b| = |a| \cdot |-b| \\
a \cdot b = a \cdot b
\]

Case 2
Let \( x = -a \) and \( y = b \)

\[
|a \cdot b| = |-a| \cdot |b| \\
a \cdot b = a \cdot b
\]

Case 3
Let \( x = a \) and \( y = b \)

\[
|a \cdot b| = |a| \cdot |b| \\
a \cdot b = a \cdot b
\]

Response 2.2:

RTP: \( |x \cdot y| = |x| \cdot |y| \)

Proof:

\[
\Rightarrow (xy)^2 = x^2 \cdot y^2 \quad (\text{square both sides and get rid of the absolute value}) \\
\Rightarrow \sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2} \quad (\text{take square root to get rid of squared}) \\
\Rightarrow xy = \sqrt{x^2} \cdot \sqrt{y^2} \\
\Rightarrow xy = x \cdot y \\
\checkmark
\]
**Response 2.3:**

*Given:* \( x, y \in \mathbb{R} \)

*RTP:* \( |x \cdot y| = |x| \cdot |y| \)

**Proof:** When you multiply \( x \) and \( y \), depending on what they are equal to, you may get a positive or negative answer. The absolute value of \( x \cdot y \) will ensure that the answer is positive.

Example:

\[
\begin{align*}
|5 \cdot 6| &= 30 ; |5| \cdot |6| = 30 \\
|-5 \cdot -6| &= 30 ; |-5| \cdot |-6| = 30 \\
|5 \cdot -6| &= 30 ; |5| \cdot |-6| = 30 \\
|-5 \cdot 6| &= 30 ; |-5| \cdot |6| = 30
\end{align*}
\]

When you put absolute value brackets around \( x \) and \( y \) separately, this makes both \( x \) and \( y \) positive factors, which must result in the same positive value that \( |x \cdot y| \) gives you.
3. (a) Prove that \( n^3 - 3n^2 - 9 \geq 0 \) for \( n \geq 6 \), \( n \in N \).

(b) Does this inequality hold for \( n > 6 \), \( n \in N \)? Why?

(c) Does this inequality hold for \( n \geq 10 \), \( n \in N \)? Why?

(d) Does this inequality hold for \( n \geq 4 \), \( n \in N \)? Why?

(e) Does this inequality hold for \( n \geq 2 \), \( n \in N \)? Why?

Part (a):

Given: \( n \geq 6 \), \( n \in N \)

RTP: \( n^3 - 3n^2 - 9 \geq 0 \)

A Proof:

\[
\begin{align*}
n^3 &= n \cdot n^2 \quad (\text{follows from the definition of a power of } n) \\
&\Rightarrow n^3 - 3n^2 - 9 = n \cdot n^2 - 3n^2 - 9 = (n - 3) \cdot n^2 - 9 \\
&\quad \text{given} \\
&\quad n \geq 6 \\
&\downarrow \\
n - 3 &\geq 6 - 3 = 3 \\
&\quad , \quad n^2 \geq 36 \\
&\downarrow \\
(n - 3) \cdot n^2 - 9 &\geq 3 \cdot 36 - 9 \geq 0 \\
&\downarrow \\
n^3 - 3n^2 - 9 &\geq 0 \\
&\quad \text{Q.E.D.}
\end{align*}
\]

Part (b):

Yes. \( n > 6 \), \( n \in N \) is included in \( n \geq 6 \), \( n \in N \), and we proved the inequality for \( n \geq 6 \), \( n \in N \).

Part (c):

Yes. \( n \geq 10 \), \( n \in N \) is included in \( n \geq 6 \), \( n \in N \), and we proved the inequality for \( n \geq 6 \), \( n \in N \).

Part (d):

Yes. For \( n = 4 \), \( n^3 - 3n^2 - 9 = 7 \geq 0 \) and for \( n = 5 \), \( n^3 - 3n^2 - 9 = 41 \geq 0 \).

We proved the inequality for \( n \geq 6 \), \( n \in N \) and showed that it holds for \( n = 4 \) and for \( n = 5 \), thus it holds for \( n \geq 4 \), \( n \in N \).

Part (e):

No. For \( n = 3 \) the inequality does not hold: \( n^3 - 3n^2 - 9 = -9 < 0 \).
SAMPLE RESPONSES FOR PROBLEM 3:

Response 3.1:
(a) Let \( n = 6 \).
\[
6^3 - 3 \cdot 6^2 - 9 \geq 0
\]
\[
99 \geq 0
\]
(c) Let \( n = 10 \).
\[
10^3 - 3 \cdot 10^2 - 9 \geq 0
\]
\[
691 \geq 0
\]
(d) Let \( n = 4 \).
\[
4^3 - 3 \cdot 4^2 - 9 \geq 0
\]
\[
7 \geq 0
\]

Response 3.2:
(a) Given: \( n \geq 6 \).
Prove: \( n^3 - 3n^2 - 9 \geq 0 \)
\[
n^3 - 3n^2 \geq 9 \text{ (add 9 to both sides)}
\]
\[
n^2(n - 3) \geq 9 \text{ (factor out } n^2\text{)}
\]
\[
6^2(6 - 3) \geq 9 \text{ (substitute 6 for } n\text{, because 6 is the lowest possible number for } n, \text{ so if } n \geq 6 \text{, then any number above 6 will be } \geq \text{ also)}
\]
\[
36 \cdot (3) \geq 9
\]
\[
\checkmark
\]
(c) Yes, because the inequality is true when \( n \geq 6 \), so it will only get bigger, the larger \( n \) is.
Response 3.3:

(a) \( n^3 - 3n^2 - 9 \geq 0 \) for \( n \geq 6 \).

Proving by using the contrapositive of the statement:

The contrapositive is \( n < 6 \) for \( n^3 - 3n^2 - 9 < 0 \).

\[
\begin{align*}
& n < 6, \ n \in N \ \Rightarrow \ n \cdot n^2 < 6n^2 \ \Rightarrow \ n^3 < 6n^2 \ \Rightarrow \ n^3 - 6n^2 < 0 \\
& \Rightarrow n^3 - 6n^2 + 3n^2 < 3n^2 \ \Rightarrow \ n^3 - 3n^2 < 3n^2 \ \Rightarrow \ n^3 - 3n^2 - 9 < 3n^2 - 9 \\
& \Rightarrow n^3 - 3n^2 - 9 < 3(n^2 - 1) \ \Rightarrow \ n^3 - 3n^2 - 9 < 3(n-1)(n+1)
\end{align*}
\]

(d) For \( n \geq 4 \)?

No, the inequality doesn’t hold for \( n \geq 4 \) because 5 is not in the original statement. However, values for \( n \geq 4 \) will still make the inequality greater than 0.