Please provide complete and well-written solutions to the following exercises.

Due February 20, at the beginning of class.

Assignment 7

Exercise 1. Integrate the function \( f(x, y, z) = yz \) over the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies above the cone \( z = \sqrt{x^2 + y^2} \).

Exercise 2. Let \( f(x, y, z) = x^2 \). Compute \( \iint_S f \, dS \) where \( S \) is the surface \( x^2 + y^2 + z^2 = 1 \) with \( x, y, z \geq 0 \).

Exercise 3. Find the surface area of the part of the cone \( z^2 = x^2 + y^2 \) between the planes \( z = 1 \) and \( z = 4 \).

Exercise 4. Let \( a, b, c \) be positive constants. An ice cream cone is defined as the surface \( z = a\sqrt{x^2 + y^2} \) where \( z \leq b \). Suppose the ice cream cone has surface area \( c \). Find the ice cream cone of fixed surface area \( c \) and with maximum volume. (This way, you get to eat the most ice cream with the least amount of material.)

Exercise 5. Let \((x, y, z)\) be a point in Euclidean space \( \mathbb{R}^3 \). Let \( G, m \) be constants. Let \( S \) denote the sphere of radius \( R \) centered at the origin. Let \( dS \) denote the surface area element of \( S \) with respect to variables \( a, b, c \). Define the following function, which is the gravitational potential of \( S \) at the point \((x, y, z)\).

\[
V(x, y, z) = -\frac{Gm}{4\pi R^2} \iint_S \frac{dS}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}
\]

- Using a symmetry argument, show that \( V(x, y, z) \) only depends on \( ||(x, y, z)|| \). That is, if \( \Phi: \mathbb{R}^3 \to \mathbb{R}^3 \) is a rotation, then \( V(x, y, z) = V(\Phi(x, y, z)) \). (Hint: it may be helpful to write \( (a, b, c) = \Phi(\Phi^{-1}(a, b, c)) \), and then to use \( ||\Phi(d, e, f)|| = ||(d, e, f)|| \), and also use \( \Phi((a, b, c) + (d, e, f)) = \Phi(a, b, c) + \Phi(d, e, f) \). Then, use that \( dS \) does not change when we apply the rotation \( \Phi^{-1} \). That is, if \( f(a, b, c) \) is a function, you may assume that \( \iint_S f(a, b, c) \, dS = \iint_S f(\Phi^{-1}(a, b, c)) \, dS \). So, to compute \( V \) at any point, it suffices to compute \( V(0, 0, r) \) for any \( r \geq 0 \). That is, we have shown that \( V(x, y, z) = V(0, 0, ||(x, y, z)||) \).
- Let \( r \geq 0 \). Using spherical coordinates, show that

\[
V(0, 0, r) = -\frac{Gm}{4\pi} \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \sin \phi \, d\theta \, d\phi \frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos \phi}}
\]

- Using the substitution \( u = R^2 + r^2 - 2Rr \cos \phi \), show that

\[
V(0, 0, r) = -\frac{mG}{2Rr} (|R + r| - |R - r|).
\]
Verify that \( V \) satisfies the following formula

\[
V(x, y, z) = \begin{cases} 
-\frac{Gm}{||(x, y, z)||}, & \text{if } ||(x, y, z)|| > R \\
-\frac{Gm}{R}, & \text{if } ||(x, y, z)|| < R
\end{cases}
\]

In particular, a hollow sphere exerts no gravitational force inside the sphere.

**Exercise 6.** Let \( F(x, y, z) = (x, y, z) \) be a vector field. Let \( a \) be a real number. Compute the flux \( \iint_S F \cdot e_n \, dS \) outward through the surface \( S \) where \( x^2 + y^2 = 1 \) and \( 0 \leq z \leq a \).

**Exercise 7.** Find the flux of the vector field \( F(x, y, z) = (xze^y, -xze^y, z) \) through the part of the plane \( x + y + z = 1 \) that lies in the first octant, where the flux is oriented downwards. (That is, you should choose the normal vector with negative \( z \) component.)

**Exercise 8.** Let \( F(x, y, z) = (x, y, e^z) \) be a vector field. Compute \( \iint_S F \cdot e_n \, dS \) where \( S \) is the surface \( x^2 + y^2 = 9 \), \( 1 \leq z \leq 4 \), and \( e_n \) denotes the outward pointing unit normal vector to \( S \).