Please provide complete and well-written solutions to the following exercises.

Assignment 6

No due date, but the quiz in Week 6 in the discussion section (on November 3rd or 5th) will be based upon this homework.

Exercise 1. Let \( f(x, y) = x^2 + y^2 \). Compute the partial derivatives: \( f_{xx}, f_{xy}, f_{yx}, f_{yy} \).

Exercise 2. Let \( f(u, v, w, x, y, z) = u^2/v + vx + yz + \sin(xw) \). Compute the partial derivatives: \( f_{uv}, f_{wz}, f_{xyz} \).

Exercise 3. Consider the following function \( f : \mathbb{R}^2 \to \mathbb{R} \).

\[
f(x, t) = \frac{1}{\sqrt{t}}e^{-x^2/(4t)}, \quad t > 0.
\]

Show that \( f \) satisfies the heat equation (for one spatial dimension \( x \)):

\[
\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.
\]

The function \( f \) represents a single point of heat emanating through an infinite rod (the \( x \)-axis) as time passes (as \( t \) increases, \( t \geq 0 \)). The heat equation roughly says that the rate of change of heat \( f \) at the point \( x \) and at time \( t \) is equal to the average difference between the current heat at \( x \), and the neighbors of \( x \). The quantity \( \partial f/\partial t \) is the rate of change of heat, while the second derivative on the right is perhaps better understood using the second-difference quotient:

\[
\frac{\partial^2 f}{\partial x^2} = \lim_{h \to 0} \frac{f(x - h, t) - 2f(x, t) + f(x + h, t)}{h^2}.
\]

Exercise 4. Consider a function \( f(x, y, t) \) of three variables. The heat equation for two spatial dimensions \( x, y \) says

\[
\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.
\]

This equation can be interpreted in a similar way to the previous equation. As time goes to infinity, eventually the heat reaches an equilibrium, i.e. the heat does not change anymore, so \( \partial f/\partial t \) goes to zero as \( t \to \infty \). So, as \( t \to \infty \), the heat equation will say that \( f_{xx} + f_{yy} = 0 \). We define \( f_{xx} + f_{yy} \) to be the Laplacian \( \Delta \). That is, given a function \( g(x, y) \) of two variables, define

\[
\Delta g(x, y) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}.
\]

A function \( g \) satisfying \( \Delta g = 0 \) is called harmonic. Understanding harmonic functions allows us to understand equilibrium configurations of heat. Show that the following functions are harmonic:
• \( g(x, y) = x \).
• \( g(x, y) = \tan^{-1}(y/x) \).
• \( g(x, y) = \ln(x^2 + y^2) \).

**Exercise 5.** Consider the following function \( f : \mathbb{R}^2 \to \mathbb{R} \):

\[
f(x, y) = \begin{cases} 
\frac{x^3 y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\
0, & \text{if } (x, y) = (0, 0)
\end{cases}
\]

Show that \( \frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0) \). (You will need to use the definition of the derivative itself, using limits.) So, it is not always true that \( f_{xy} = f_{yx} \).