Please provide complete and well-written solutions to the following exercises.

No due date, but the quiz in Week 2 in the discussion section (on October 6th or 8th) will be based upon this homework.

Assignment 2

Exercise 1. Define

\[ H(x) = \begin{cases} 
0, & \text{if } x < 0 \\
1, & \text{if } x \geq 0
\end{cases} \]

Explain in your own words why \( \lim_{x \to 0} H(x) \) does not exist.

Exercise 2. Find two functions \( f, g \) such that \( \lim_{x \to a} f(x) \) does not exist, \( \lim_{x \to a} g(x) \) does not exist, but such that \( \lim_{x \to a} (f(x) + g(x)) \) does exist.

Exercise 3. Evaluate the following limit and justify each step by indicating the appropriate limit law.

\[ \lim_{u \to -2} \sqrt{u^4 + 3u + 6} \]

Exercise 4. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

\[ \lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right) \]

Exercise 5. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

\[ \lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1} \]

Exercise 6. Is there a real number \( a \) such that the following limit exists?

\[ \lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} \]

If so, find the value of \( a \) and the value of the limit.

Exercise 7. Are the following statements true or false?

(a) If \( \lim_{x \to 5} f(x) = 0 \) and \( \lim_{x \to 5} g(x) = 0 \), then \( \lim_{x \to 5} \frac{f(x)}{g(x)} \) does not exist.

(b) If \( x \) is a real number, then \( \sqrt{x^2} = x \)

(c) If \( \lim_{x \to 5} f(x) = 2 \) and \( \lim_{x \to 5} g(x) = 0 \), then \( \lim_{x \to 5} \frac{f(x)}{g(x)} \) does not exist.

(d) If \( f \) is continuous at 5 and \( f(5) = 2 \), then \( \lim_{x \to 2} f(4x^2 - 11) = 2 \).

(e) If \( f(x) > 1 \) for all \( x \neq 0 \) and \( \lim_{x \to 0} f(x) \) exists, then \( \lim_{x \to 0} f(x) > 1 \).
Exercise 8. Fix $x \in \mathbb{R}$, and let $f(x) = x^2$. Calculate the following limit
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]
The fraction $(f(x+h) - f(x))/h$ is known as a difference quotient. The limit of this difference quotient will come up again later in the course.

Exercise 9. Let $f, g: \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$. Is it always true that $\lim_{x \to a} (f(x) + g(x)) = (\lim_{x \to a} f(x)) + (\lim_{x \to a} g(x))$?

Exercise 10. Find all values of $a$ and $b$ such that the following function is continuous:
\[
f(x) = \begin{cases} 
ax - b & x \leq -1 \\
2x^2 + 3ax + b & -1 < x \leq 1 \\
4 & x > 1 
\end{cases}
\]

Exercise 11. For what values of $x$ is the following function continuous: $g(x) = (\sin(3x^5 + 10))^{1/3}$. (Hint: treat each function as a composite function, and look at the domain of each part.)

Exercise 12. Find the following limit
\[
\lim_{x \to 0} \frac{\sin(3x) \sin(5x)}{x^2}.
\]

Exercise 13. Draw the following set and describe it in words: the set of all points $(x, y)$ in the plane such that
\[
\lim_{t \to \infty} (|x|^t + |y|^t) < 4.
\]