Please provide complete and well-written solutions to the following exercises.

Due January 14th, in the discussion section.

**Homework 1**

**Exercise 1.** Let $A, B, C$ be sets in a universe $\Omega$. Using the definitions of intersection, union and complement, prove properties (ii) and (iii) below.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(iii) $(A^c)^c = A$.

(Hint: to prove property (ii), it may be helpful to first draw a Venn diagram of $A, B, C$. Now, let $x \in \Omega$. Consider where $x$ could possibly be with respect to $A, B, C$. For example, we could have $x \in A$, $x \notin B$, $x \in C$. We could also have $x \in A$, $x \in B$, $x \notin C$. And so on. In total, there should be $2^3 = 8$ possibilities for the location of $x$, with respect to $A, B, C$. Construct a truth table which considers all eight such possibilities for each side of the purported equality $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.)

**Exercise 2.** Let $A_1, A_2, \ldots$ be sets in some universe $\Omega$. Prove that $(\bigcap_{i=1}^{\infty} A_i)^c = \bigcup_{i=1}^{\infty} A_i^c$.

**Exercise 3.** Let $A_1, A_2, \ldots$ be sets in some universe $\Omega$. Let $B \subseteq \Omega$. Prove:

$$B \cap \left( \bigcup_{k=1}^{\infty} A_k \right) = \bigcup_{k=1}^{\infty} (A_k \cap B).$$

**Exercise 4** (Discrete Uniform Probability Law). Let $n$ be a positive integer. Suppose we are given a finite universe $\Omega$ with exactly $n$ elements. Let $A \subseteq \Omega$. Define $P(A)$ such that $P(A)$ is the number of elements of $A$, divided by $n$. Verify that $P$ is a probability law. This probability law is referred to as the uniform probability law on $\Omega$, since each element of $\Omega$ has the same probability.

**Exercise 5.** Let $\Omega = \mathbb{R}^2$. Let $A \subseteq \Omega$. Define a probability law $P$ on $\Omega$ so that

$$P(A) = \frac{1}{2\pi} \int_{A} e^{-(x^2+y^2)/2} dxdy.$$

We can think of $P$ as defining the (random) position of a dart, thrown at an infinite dart board. That is, if $A \subseteq \Omega$, then $P(A)$ is the probability that the dart will land in the set $A$.

Verify that Axiom (iii) holds for $P$. That is, verify that $P(\Omega) = 1$. Then, compute the probability that a dart hits a circular board $A$, where $A = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \leq 1\}$.

**Exercise 6.** Let $A, B$ be subsets of a sample space $\Omega$. Prove the following things:

- $A = (A \cap B) \cup (A \cap B^c)$.
- $A = (A \setminus B) \cup (A \cap B)$, and $(A \setminus B) \cap (A \cap B) = \emptyset$. 

• $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$, and the three sets $(A \setminus B), (B \setminus A), (A \cap B)$ are all disjoint. That is, any two of these sets are disjoint.

**Exercise 7.** Let $\Omega$ be a sample space and let $\mathbf{P}$ be a probability law on $\Omega$. Let $A, B, C \subseteq \Omega$. Prove the following things:

- $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$.
- $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$.

(Although the book suggests otherwise, a Venn diagram alone is not a rigorous proof. As in Exercise 1, a truth table allows us to rigorously reason about the information contained in a Venn diagram. Though, there are ways to do the problem while not directly using a truth table.)

**Exercise 8.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that

$$\bigcup_{y \in \mathbb{R}} \{ x \in \mathbb{R} : f(x) = y \} = \mathbb{R}.$$ 

Also, show that the union on the left is disjoint. That is, if $y_1 \neq y_2$ and $y_1, y_2 \in \mathbb{R}$, then

$$\{ x \in \mathbb{R} : f(x) = y_1 \} \cap \{ x \in \mathbb{R} : f(x) = y_2 \} = \emptyset.$$