Please provide complete and well-written solutions to the following exercises.

Due May 31, in the discussion section.

**Homework 8**

**Exercise 1.** Prove the following Lemma from the notes: The set of functions \( W_S \) for \( S \subseteq \{\ldots \} \) is an orthonormal basis for the space of functions from \( (-1,1)^n \rightarrow \mathbb{R} \), with respect to the inner product defined in the notes. (When we write \( S \subseteq \{1,\ldots,n\} \), we include the empty set \( \emptyset \) as a subset of \( \{1,\ldots,n\} \).) (Also, for any \( x \in (-1,1)^n \), \( W_S(x) = \prod_{i \in S} x_i \).)

**Exercise 2.** Let \( f: \{-1,1\}^2 \rightarrow \{-1,1\} \) such that \( f(x) = 1 \) for all \( x \in \{-1,1\}^2 \). Compute \( \hat{f}(S) \) for all \( S \subseteq \{1,2\} \).

Let \( f: \{-1,1\}^3 \rightarrow \{-1,1\} \) such that \( f(x_1, x_2, x_3) = \text{sign}(x_1 + x_2 + x_3) \) for all \( (x_1, x_2, x_3) \in \{-1,1\}^3 \). Compute \( \hat{f}(S) \) for all \( S \subseteq \{1,2,3\} \). The function \( f \) is called a **majority function**.

**Exercise 3.** Let \( f: \{-1,1\}^3 \rightarrow \{-1,1\} \) such that \( f(x_1, x_2, x_3) = \text{sign}(x_1 + x_2 + x_3) \) for all \( (x_1, x_2, x_3) \in \{-1,1\}^3 \). In the previous exercise, we computed \( \hat{f}(S) \) for all \( S \subseteq \{1,2,3\} \). The function \( f \) is called a **majority function**. Compute the noise stability of \( f \), for any \( \rho \in (-1,1) \).

Let \( n \) be a positive odd integer. The majority function for \( n \) voters can be written as \( f(x_1,\ldots,x_n) = \text{sign}(x_1 + \cdots + x_n) \), where \( x_1,\ldots,x_n \in \{-1,1\} \) and \( f: \{-1,1\}^n \rightarrow \{-1,1\} \). In the limit as \( n \rightarrow \infty \), the noise stability of the majority function approaches a limiting value. (We implicitly used this fact in stating the Majority is Stablest Theorem.) You will compute this limiting value \( A \) in the following way. We have \( A = 4B - 1 \), where \( B \) is defined below.

Let \( z_1, z_2 \) be vectors of unit length in \( \mathbb{R}^2 \). Let \( \rho \in (-1,1) \). Let \( \cdot \) denote the standard inner product of vectors in \( \mathbb{R}^2 \). Assume that \( z_1 \cdot z_2 = \rho \). Let \( C \subseteq \mathbb{R}^2 \) be the set such that \( C = \{(x,y) \in \mathbb{R}^2: (x,y) \cdot z_1 \geq 0 \text{ and } (x,y) \cdot z_2 \geq 0 \} \).

Then \( B = \int_C \int e^{-(x^2 + y^2)/2} \frac{\text{d}x \text{d}y}{2\pi} \). Compute the value of \( A \). (You should get a relatively simple quantity involving an inverse trigonometric function.)

**Exercise 4.** Let \( f \) denote the majority function for \( n \) voters. In class, we showed that \( I_i(f) \approx 1/\sqrt{n} \) for all \( i \in \{1,\ldots,n\} \). Explain why we can interpret this calculation as saying: your influence in a majority election is a lot more than \( 1/n \), so you should vote. On the other hand, give reasons why the influence calculation may not accurately reflect your actual influence in a majority election. (If you are thinking of elections in the US, feel free to consider or ignore the electoral college system.)
Exercise 5. Let \( n \) be a positive integer. Let \( f, g: \{-1, 1\}^n \to \{-1, 1\} \). Let \( a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{R} \). Let \( x = (x_1, \ldots, x_n) \in \{-1, 1\}^n \). For any \( x \in \{-1, 1\}^n \), define \( L_f(x) = a_0 + \sum_{i=1}^n a_i x_i \), \( L_g(x) = b_0 + \sum_{i=1}^n b_i x_i \). Assume that \( L_f(x) \neq 0 \) and \( L_g(x) \neq 0 \) for all \( x \in \{-1, 1\}^n \). Assume also that \( f(x) = \text{sign}(L_f(x)) \) and \( g(x) = \text{sign}(L_g(x)) \) for all \( x \in \{-1, 1\}^n \).

Assume that \( \hat{f}(S) = \hat{g}(S) \) for all \( S \subseteq \{1, \ldots, n\} \) with \( |S| \leq 1 \). Prove that \( f = g \). (Hint: what does the Plancherel Theorem say about \( \langle f, L_f \rangle \)?) How does this quantity compare to \( \langle g, L_f \rangle \)? Also, note that \( f(x)L_f(x) = |L_f(x)| \geq g(x)L_f(x) \) for any \( x \in \{-1, 1\}^n \).

(Recall \( \text{sign}(t) = 1 \) if \( t > 0 \) and \( \text{sign}(t) = -1 \) if \( t < 0 \).)

Exercise 6. Let \( n \) be a positive integer. Show that there is a one-to-one correspondence (or a bijection) between the set of functions \( f \) where \( f: \{-1, 1\}^n \to \mathbb{R} \), and the set of functions \( g \) where \( g: 2^\{1,2,\ldots,n\} \to \mathbb{R} \). For example, you could identify a subset \( S \subseteq \{1, \ldots, n\} \) with the element \( x = (x_1, \ldots, x_n) \in \{-1, 1\}^n \) where, for all \( i \in \{1, \ldots, n\} \), we have \( x_i = 1 \) if \( i \in S \), and \( x_i = -1 \) if \( i \notin S \).

Let \( i, j \in \{1, \ldots, n\} \) and let \( x \in \{-1, 1\}^n \). Let \( S(x) = \{ j \in \{1, \ldots, n\} : x_j = 1 \} \). Using this one-to-one correspondence, show that the \( i \)th Shapley value of \( f: \{-1, 1\}^n \to \{-1, 1\} \) can be written as

\[
\phi_i(f) = \sum_{x \in \{-1, 1\}^n : x_i = -1} \frac{|S(x)|!(n-|S(x)|-1)!}{n!} (f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - f(x)).
\]

So, \( \phi_i(f) \) is similar to, but distinct from, \( I_i(f) \). On the other hand, the \( i \)th Banzhaf power index is essentially identical to \( I_i(f) \). That is, if we define

\[
B_i(f) = \sum_{x \in \{-1, 1\}^n} \frac{\left| f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) - f(x) \right|}{2},
\]

Then \( B_i(f)/\sum_{j=1}^n B_j(f) \) is the \( i \)th Banzhaf power index of \( f \).

Exercise 7. Let \( f: \{-1, 1\}^n \to \{-1, 1\} \). Assume that \( \hat{f}(S) = 0 \) whenever \( S \subseteq \{1, \ldots, n\} \) and \( |S| \neq 1 \). Show that there exists \( i \in \{1, \ldots, n\} \) such that \( f(x) = f(x_1, \ldots, x_n) = x_i \) for all \( x \in \{-1, 1\}^n \), or \( f(x) = -x_i \) for all \( x \in \{-1, 1\}^n \). (This exercise therefore completes the proof of Arrow’s Theorem.)