Mid-Term 1

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may not use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- If you use a theorem or proposition from class or the notes or the book you must indicate this and explain why the theorem may be applied. It is okay to just say, “by some theorem/proposition from class.”
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck!"
1. (a) (5 points) Give an example of a game mentioned in class, or the notes, or any course textbook, such that: the first player has a winning strategy. (You only need to mention the game; you do not need to prove anything.)

(b) (5 points) Give an example of a game mentioned in class, or the notes, or any course textbook, such that: the second player has a winning strategy. (You only need to mention the game; you do not need to prove anything.)

(c) (5 points) Give an example of a game mentioned in class, or the notes, or any course textbook, such that: both players have a strategy guaranteeing at least a draw. (You only need to mention the game; you do not need to prove anything.)
2. (20 points) Consider the game of Nim, where the game starts with four piles of chips. These piles have 1, 5, 3 and 15 chips, respectively. Which player has a winning strategy from this position, the first player, or the second? Describe a winning first move. Prove that this move is a winning first move.
3. (25 points) Describe the optimal strategies for both players for the two-person zero-sum game described by the payoff matrix below. That is, at the optimal strategy, with what probability does player I play C, with what probability does player I play D, with what probability does player II play A, with what probability does player II play B?

\[
\begin{array}{c|cc}
\text{Player I} & \text{A} & \text{B} \\
\hline
\text{C} & 0 & 1 \\
\text{D} & 2 & 1 \\
\end{array}
\]

Prove that these strategies are optimal.
4. (15 points) Let $Y$ be a random variable such that: $Y = 2$ with probability $1/2$, $Y = 3$ with probability $1/2$.

Let $Z$ be a random variable such that: $Z = 1$ with probability $1/2$ and $Z = 2$ with probability $1/2$. Assume that $Z$ and $Y$ are independent. What is the probability that: $Y = 3$ and $Z = 2$? What is the expected value of $Y \cdot Z$?
5. (15 points) Explicitly define some function \( f: [0, 1] \times [0, 1] \rightarrow \mathbb{R} \) such that

\[
\min_{y \in [0,1]} \max_{x \in [0,1]} f(x, y) \neq \max_{x \in [0,1]} \min_{y \in [0,1]} f(x, y).
\]

Prove that the function \( f \) satisfies this property.
(Scratch paper)