1. Show that there is a non-empty $\Sigma^1_1$ set which contains no element of $L_{\omega_1^{ck}}$.

Let $C = \{ x : x \in L_{\omega_1^{ck}} \}$ in exercises 2-4 below.

2. Show that $C$ is $\Pi^1_1$.

3. Show that if $B \subset 2^\omega$ is $\Pi^1_1$ and contains no perfect set, then $B \subset C$. (Hint: Let $T \subset 2^{<\omega} \times 2^{<\omega}$ be a recursive tree with $B = \{ x : \forall y \in \omega \exists n ((x|_n, y|_n) \notin T) \}$. Let $(s_n)_{n \in \omega}$ enumerate $\omega^{<\omega}$ with each $\lh(s_n) \leq n$. For $\alpha < \omega_1^{ck}$ we can let $S_\alpha$ be the set of $(t, f) \in 2^{<\omega} \times \alpha^{<\omega}$ such that $\lh(t) = \lh(f)$ and for all $n, m < \lh(f)$ with

\[(t|_{\lh(s_n)}, s_n) \in T, \]
\[(t|_{\lh(s_m)}, s_m) \in T, \]
\[s_n \subset s_m, \]
\[s_n \neq s_m, \]

we have $f(n) > f(m)$. First argue that $x \in B$ if and only if for some $\alpha \in \omega_1^{ck}$ we have that there exists $f \in \alpha^{<\omega}$ with

\[(x|_n, f|_n) \in S_\alpha \]

at every $n \in \omega$. Then adapt the perfect set theorem for $\Sigma^1_1$ sets to argue that if $L_\delta \models \text{KP}$ then at every $\alpha < \delta$ we have either

\[\{ x : \exists f \forall n ((x|_n, f|_n) \in S_\alpha) \}

contains a perfect set or it is included in $L_\delta$.)

4. (i) For $f : 2^\omega \to C$ continuous and $\Delta^1_1(z)$, adapt the proof of Kriesel uniformization to show that there is a $\Delta^1_1(z)$ function $g : 2^\omega \to \omega$ such that at every $x$ if $\alpha = |W^x_{g(x)}|$ then $f(x) \in L_\alpha$. (Here the intention is that $g$ be $\Delta^1_1(z)$ as a subset of $2^{<\omega} \times \omega$.)

(ii) Show that $C$ does not contain a perfect set. (Hint: Use (i) to show that if there was a continuous, injective $f : 2^\omega \to C$ then we would get a $\Sigma^1_1(z)$ subset of $\{(x, e) : W^x_e \text{ is a well ordering } \}$ with the set of $|W^x_e|$ unbounded in $\omega_1$.)