Math 223d, Spring 2004, Strong Choquet Spaces

**Definition** The *strong Choquet game* on a space $X$ consists of players I and II alternately playing $U_n, x_n$ by I and then $V_n$ by II,

$$
U_1, x_1 \quad U_2, x_2 \quad \ldots \\
V_1 \quad V_2 \quad \ldots
$$

with each $U_n$ an open set containing $x_n$ and then $V_n$ an open subset of $U_n$ again containing $x_n$, and then $U_{n+1}$ an open subset of $V_n$, and so on.

The game is an infinite game, which is played out for infinitely many turns. After time itself has come to an end, we evaluate a particular play and say that II *wins* if the intersection

$$
\bigcap_{n \in \mathbb{N}} V_n
$$

is non-empty. We say that II has a winning strategy if there is some function $\sigma$ from finite plays of the form

$$p = \begin{pmatrix}
U_1, x_1 & U_2, x_2 & \ldots & U_n, x_n \\
V_1 & V_2 & \ldots & V_{n-1}
\end{pmatrix}
$$

to legal responses $V_n = \sigma(p)$ such that in any infinite play in which II always follows the strategy we will have $\bigcap_n V_n \neq \emptyset$.

We denote this game by $G^*_X$ and say that $X$ is a *strong Choquet space* if II has a winning strategy.

It is easily seen that any Polish space is strong Choquet: II simply makes sure at stage $n$ to play $V_n$ containing $x_n$ with diameter less than $2^{-n}$ and closure included in $U_n$. It follows then as in some of the arguments involving Lusin schemes that $\bigcap_n V_n \neq \emptyset$.

Much more exciting is the existence of a converse.

**Theorem** Any separable, metrizable, strong Choquet space is Polish.

**Proof** Let $X$ be a separable, strong Choquet space with compatible metric $d$. Let $\hat{X}$ be its Cauchy completion.

**Notation** For $U \subset X$ open in $X$, let

$$
\hat{U} = (\overline{U^X})^\circ \hat{X},
$$

the interior of the closure of $U$ as *calculated inside the space* $\hat{X}$ – that is to say we first take the closure of $U$ in $\hat{X}$, and then the interior inside $\hat{X}$.

(Warning: Kechris does not exactly use this notation.)

Thus $\hat{U}$ will be a $\hat{X}$-open set having $U$ as a dense subset. Note that while $U$ is open in $X$, we cannot expect that it will continue to be open in $\hat{X}$, any more than the a set which is open in the rationals will be open inside all of the reals.

We fix $\sigma$ a winning strategy for II in $G^*_X$. We seek to show that $X$ is a $G_\delta$ subset$^1$ of $\hat{X}$, which will do it since $G_\delta$ subsets of Polish spaces are again Polish.

We are going to produce an array of finite partial plays, $(p_s)_{s \in \mathbb{N}<\infty}$, where in each such play II’s moves are chosen according to $\sigma$, $p_s \subset p_t$ if $s \subset t$, and the length of $p_s$ is equal to the length of $s$. In particular, the first move of $p_s$ depends solely on the first value of $s$, the first two moves just on $s_{[2]}$, the restriction of $s$ to its first two terms, the first $m$ moves on first $m$ terms of $s$, and so on. Thus we are not mistreating our notation to write:

$$p_s = \begin{pmatrix}
U_{s_{[1]}, x_{s_{[1]}}} & U_{s_{[2]}, x_{s_{[2]}}} & \ldots & U_{s_{[m]}, x_{s_{[m]}}} \\
V_{s_{[1]}} & V_{s_{[2]}} & \ldots & V_{s_{[m]}}
\end{pmatrix}.
$$

We are also going to require the following three conditions:

$^1$I.e. defined by the countable intersection of open sets
(i) at each $\ell$

$$X = \bigcup_{s \in \mathbb{N}^{\ell}} V_s;$$

(ii) at each $s$, $d(U_s) < 2^{-\text{length}(s)}$

(iii) for each $\ell$ and each $\hat{x} \in \hat{X}$, there are only finitely many $s$ with $\hat{x} \in U_{s-i}^\ell$ for some $i$.

The proof that we can arrange all this is by a careful induction, moving up and dealing with all sequences of a given length in one go. Let us hold off on giving that proof and first see that if we arrange (i)-(iii) then we are indeed done.

So assuming (i), (ii), and (iii), let $G \subseteq X$ be the set of $\hat{x}$ such that at every $\ell$ there is some $s \in \mathbb{N}^\ell$ with $\hat{x} \in U_s^\ell$. In other words,

$$G = \bigcap_{\ell} \bigcup_{s \in \mathbb{N}^{\ell}} \hat{U}_s;$$

and is transparently $G^\ell$. Clearly every $x \in X$ is in $G$, since at every $\ell$ there will be some $s \in \mathbb{N}^\ell$ with $x \in V_s \subseteq U_s \subseteq \hat{U}_s$ by (i). Conversely if $\hat{x}$ is in $G$, then by (iii) we can form the finite branching tree $T_\ell$ consisting of all $s \in \mathbb{N}^{<\mathbb{N}}$ for which there exists $i$ with $\hat{x} \in U_{s-i}^\ell$. Since $\hat{U}_s \subseteq \hat{U}_t$ for $s \supseteq t$, it follows from $\hat{x} \in G$ that $T_\ell$ is infinite, and hence by König there is an infinite branch through it. Thus we get $f \in \mathbb{N}^\ell$ with $\hat{x} \in \bigcap_n U_{f[n]}^\ell$. But since we are making all $\Pi$'s moves according to a $G^K$ winning strategy, it follows that $\bigcap_n V_{f[n]}^\ell = \bigcap U_{f[n]}^\ell$ is non-empty; let $x \in X$ be some point in that infinite intersection. Then it follows from (ii) that

$$d(x, \hat{x}) \leq 2^{-t}$$

all $\ell \in \mathbb{N}$, and so in fact these points are equal, as required to show $\hat{x} \in X$.

So this only leaves us with the work of establishing that we can indeed build an array of plays as required. For this purpose, let us assume inductively that we have built $(p_s)_{s \in \mathbb{N}^{\ell}}$ as required, each

$$p_s = \begin{pmatrix} U_{s[1], x_{s[1]}} & U_{s[2], x_{s[2]}} & \cdots & U_{s, x_s} \\ V_{s[1]} & V_{s[2]} & \cdots & V_s \end{pmatrix}.$$

**Claim:** There is a collection of open sets $(O_s)_{s \in \mathbb{N}^{\ell}}$ such that each $O_s \subseteq V_s$ and

(a) $\bigcup O_s = \bigcup V_s$

(b) each $\hat{x} \in \hat{X}$ appears in only finitely many $O_s$'s.

**Proof of claim:** Let $(s_n)_{n \in \mathbb{N}}$ enumerate $\mathbb{N}^\ell$. Let $O_{s_n}$ be

$$\{ x \in V_{s_n} : \forall m < n (d(x, (X \setminus V_{s_m})) < \frac{1}{n}) \}.$$

This is open, since we have created $O_{s_n}$ by removing from $V_{s_n}$ with finitely many closed subsets of earlier $V_{s_m}$'s.

Note that each $x \in X$ can appear in only finitely many of these open sets: Once $x \in O_{s_n}$, we have $x \in V_{s_n}$ and in particular for some $m$ we have $d(x, (X \setminus V_{s_m})) \geq \frac{1}{m}$, and then for all $k > m$ it follows that $x \notin O_{s_k}$. Conversely, if $x \in \bigcup V_{s_n}$ then we can consider the least $m$ with $x \in V_{s_m}$, and obtain instantly that $x \in O_{s_m}$.

(Clima)\(\)

Now we choose $W_{s-i}^\ell \subseteq O_s$ for each $s \in \mathbb{N}^\ell$, $i \in \mathbb{N}$ so that $d(W_{s-i}^\ell) < 2^{-\ell-1}$ and

$$\bigcup_i W_{s-i}^\ell = O_s.$$  

For each $x \in W_{s-i}^\ell$, let $V_{s-i;x}$ be the response according to $\Pi$'s winning strategy $\sigma$ to the play

$$U_{s[1], x_{s[1]}} & U_{s[2], x_{s[2]}} & \cdots & U_{s, x_s} & W_{s-i; x}.$$  

Note that

$$\bigcup_{x \in W_{s-i}^\ell} V_{s-i;x} = W_{s-i}^\ell.$$
for the almost trivial reason that $V_{s^{-}i,x}$ must contain $x$. Thus by the separability of the space\(^2\), there is a countable collection $\{x_{s,i,j} : j \in \mathbb{N}\}$ such that
\[
\bigcup_{j \in \mathbb{N}} V_{s^{-}i,x_{s,i,j}} = W_{s^{-}i}.
\]
We let $(U_{s^{-}i,x_{s^{-}i}})_{i \in \mathbb{N}}$ be some enumeration of the countable collection $\{(W_{s^{-}i,x_{s,i,j}}) : i, j \in \mathbb{N}\}$.

We have thereby enforced (i) for the new collection $(V_{s^{-}i})_{s \in \mathcal{N}, i \in \mathbb{N}}$, since we arranged
\[
\bigcup_{i} V_{s^{-}i} = \bigcup_{i} U_{s^{-}i} = \bigcup_{i} W_{s^{-}i} = O_{s}
\]
and we previously were able to demand that
\[
\bigcup_{s \in \mathcal{N}} O_{s} = \bigcup_{s \in \mathcal{N}} V_{s} = X.
\]

We have maintained (ii), since each $U_{s^{-}i}$ is equal to some $W_{s^{-}j}$, and these were chosen to have diameter less than $2^{-l-1} = 2^{-\text{length}(s)-1}$.

We have managed to preserve (iii) since at each $\hat{x}$ there are only finitely many $s$ with $\hat{x} \in \hat{O}_{s}$, and hence only finitely many $s$ for which $\hat{x} \in U_{s^{-}i} \subset \hat{O}_{s}$.

Thus by induction, we succeed to build $(p_{s})_{s \in \mathcal{N} \cap \mathbb{N}}$, and finish as remarked earlier. \(\square\)

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\(^2\)Recall: There can be no uncountable strictly increasing sequence of open sets in a second countable space, and hence, in particular, any uncountable collection of open sets will have a countable subcollection with the same union.