Math 223d, HW8

Due date: Wednesday, June 9, start of class (11:30am).

Let \( LO = \{ x \in 2^{N \times N} : x \) defines a linear ordering of \( N \} \); let \( WO = \{ x \in LO : x \) defines a well ordering of \( N \} \). Let \( Tr \) be the space of trees on \( N < N \).

Take for granted the following two facts\(^1\): (1) For any standard Borel space \( X \) and \( \Sigma^1_1 A \subset X \) there is a Borel function \( f_A : X \rightarrow Tr \)
such that \( x \in A \) if and only if \( [f_A(x)] \neq \emptyset \). (2) There is a continuous
\[
\pi : Tr \rightarrow LO
\]
such that \([T] = \emptyset \iff f(T) = WO\), and moreover, when \([T] = \emptyset\) the well ordering \( f(T)\) is isomorphic to at least as big as \( \rho_T(\emptyset) \) (the rank of the empty sequence in the tree \( T \)). In particular, putting these two facts together we get that for any \( \Sigma^1_1 \) set \( A \subset X \) there is Borel \( g_A : X \rightarrow LO \) such that \( X \setminus A = g_A^{-1}(WO) \), which is all you will need for question 1.

For \( w \in WO \), let \( |w| \) be the unique ordinal to which it is isomorphic (as a linear ordering).

**Q1:** (a) Show that the set of pairs \((w, v) \in LO \times LO \) such that either \( w \notin WO \) or \((v, w) \in WO \) and \( |v| < |w| \) forms a \( \Sigma^1_1 \) set.

(b) Let \( B \subset WO \) be \( \Sigma^1_1 \) and \( f : X \rightarrow LO \) be Borel. Let \( C \) be the set of \( x \in f^{-1}(WO) \) for which there is some \( b \in B \) such that \( f(x) \) is isomorphic to some ordinal \( \alpha \) and \( b \) is isomorphic to some ordinal \( \beta \) and
\[
\alpha < \beta.
\]
Show that \( C \) is \( \Sigma^1_1 \).

(c) Show that if \( B \subset WO \) is \( \Sigma^1_1 \) then there is an ordinal \( \delta < \omega_1 \) such that \( |w| < \delta \) all \( w \in B \). (Hint: Use the following two facts: There exists \( A \in \Sigma^1_1 \) whose complement is not \( \Sigma^1_1 \). There exists \( g_A : X \rightarrow LO \) with \( f^{-1}(WO) \) the complement of \( A \).)

**Q2 For extra credit:** (a) Define
\[
h : Tr \times Tr \rightarrow Tr
\]
by letting \( h(T_1, T_2) \) be the set of sequences of the form
\[
(s_0, t_0, s_1, t_1, \ldots, s_n)
\]
or
\[
(s_0, t_0, \ldots, s_n, t_n),
\]
for \( s \in T_1, t \in T_2 \). Show that \( h \) is Borel.

(b) Show that \( h(T_1, T_2) \) is well founded exactly when at least one of the \( T_i \)'s is well founded, and in that case we have \( \rho_{h(T_1, T_2)}(s_0, t_0, \ldots, s_n, t_n) \) equaling the infimum of
\[
\rho_{T_1}(s_0, s_1, \ldots, s_n), \rho_{T_2}(t_0, \ldots, t_n).
\]

(c) Let \( A_1, A_2 \) be disjoint \( \Sigma^1_1 \) of \( X \); let \( g_1 = g_{A_1}, g_2 = g_{A_2} : X \rightarrow Tr \) be as above, with
\[
[g_i(x)] \neq \emptyset \iff x \in A_i.
\]
Show that we can find a Borel function \( f : X \rightarrow WO \) such that at every \( x \) we have \( f(x) \) at least equal to \( \inf\{\rho_{g_1(x)}(\emptyset), \rho_{g_2(x)}(\emptyset)\} \).

(d) Use question one to conclude that there is an ordinal \( \delta \) such that at every \( x \) one of \( \rho_{g_1(x)}(\emptyset), \rho_{g_2(x)}(\emptyset) \) is less than \( \delta \).

(e) Note that with \( \{ x \in X : \rho_{g_1(x)}(\emptyset) > \delta \} \) we obtain a Borel set including \( A_1 \) but disjoint from \( A_2 \).

At first sight this might seem like a very cumbersome and artificial way to prove the separation theorem for \( \Sigma^1_1 \) sets. However this kind of use of ordinals is a powerful method which has many other applications.

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\(^1\)To a large extent these were both implicit in the last homework set