Math 223d, HW8

Due date: Wednesday, June 2, start of class (11:30am).

Q1: (a) Let $X$ be a Polish space and $A \subseteq X$ a $\mathcal{S}^1_1$ set. Let $Tr$ be the elements of $P(\mathbb{N}^{<\mathbb{N}})$ (the collection of subsets of $\mathbb{N}^{<\mathbb{N}}$ in the topology it obtains via the canonical identification with $2^{\mathbb{N}^{<\mathbb{N}}}$) closed downwards under inclusion; in other words, $Tr$ is the space of trees on $\mathbb{N}^{<\mathbb{N}}$.

Show that there is a Borel function

$$X \rightarrow Tr$$

$$x \mapsto T_x$$

such that at any $x \in X$ we have

$$x \in A \iff [T_x] \neq \emptyset.$$

(b) Conclude that the collection of $T \in Tr$ with $[T] \neq \emptyset$ is not Borel.

(c) Show that the set of $T \in Tr$ with $[T] \neq \emptyset$ is $\mathcal{S}^1_1$.

(d) Let $LO$ be the collection of linear orderings on $\mathbb{N}$ (in the topology it inherits from being identified with a closed subset of $2^{\mathbb{N} \times \mathbb{N}}$). By considering the Kleene-Brouwer ordering on a tree, show that there is a continuous function $f: Tr \rightarrow LO$ such that $f(T)$ is a wellordering if and only if $[T] = \emptyset$.

Hence conclude that the set of illfounded orderings on $\mathbb{N}$ is not a Borel set.

(e) Show that the collection of illfounded elements of $LO$ is a $\mathcal{S}^1_1$ set.

(f) Show that in part (a), if $X$ is zero-dimensional, then we can find $x \mapsto T_x$ which is actually continuous. (Hint: By setting up the appropriate Suslin scheme, we may assume $X = [S]$ for some tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$. Assume $A = p[F]$ for some $F \subseteq \mathbb{N}^n \times \mathbb{N}^n$ close. Then we put a node $u \in \mathbb{N}^n$ into $T_x$ if and only if there is some $(y,x') \in F$ with $y \supset u$, $x' \supset x|\ell$).

Question 2: Extra credit only: The idea of the next sequence of problems is to show that the category quantifiers $\exists^* x$, $\forall^* x$ applied to Borel sets keep us inside the class of Borel.

(a) Show that if $(B_n)$ is a sequence of Borel sets in a Polish space, then $\bigcup B_n$ is comeager if and only if at every nonempty basic open $U$ there is an $n$ and a basic open nonempty $V \subseteq U$ with $B_n \cap V$ comeager in $V$.

(b) Suppose $X,Y$ Polish. Show that for $O \subseteq X \times Y$ open and $U \subseteq Y$ open, the set of $x \in X$ for which $O_x \cap U$ is comeager is a $G_\delta$ subset of $X$. (Hint: An open set is comeager if and only if it is open dense.)

Temporary notation: For $B \subseteq X \times Y$ and open $U \subseteq Y$ let $X^*(B,U)$ be the set of $x \in X$ for which $B_x \cap U$ is comeager in $U$.

(c) Show that $B$ is such that $X^*(B,U)$ is always Borel then the same holds for $X^*(X \setminus B,U)$.

(d) Show that if at each $n$ we have $B_n$ having $X^*(B_n,U)$ Borel for all open $U \subseteq Y$, then at every open $U \subseteq Y$ we likewise have $X^*(\bigcup B_n,U)$ Borel. (Here: (a) is helpful.)

(e) Conclude that if $B$ is any Borel set, then the set of $x$ for which $B_x$ is comeager is Borel. Hence, $B$ Borel implies $\{ x : \forall^* y \in Y((x,y) \in B) \}$ and $\{ x : \exists^* y \in Y((x,y) \in B) \}$ are both Borel.