**Math 223d, HW1**

**Due date:** Wednesday, April 21, start of class (11:30am).

**Q1:** (a) Show that any compact metric space has a countable dense subset. (Hint: it follows from one of the standard characterizations that a compact metric space can be covered by finitely many balls of radius $\varepsilon$ for every positive $\varepsilon$. By doing this at every such rational $\varepsilon$ we obtain a countable dense subset.)

(b) Conclude that every compact metric space has size at most $2^\aleph_0$. (Hint: The number of Cauchy sequences from our countable dense subset is at most $2^\aleph_0$.)

**Q2:** (a) Construct a tree $T \subset \mathbb{N}^\mathbb{N}$ for which the rank in $T$ of the sequence $s = (2, 3)$, that is to say

$$\rho_T(s),$$

equals (exactly)

$$\omega + 2.$$

(b) Show that no such $T$ can be finite branching.

**Remark:** Just off on a tangent from question 1, it has come up several times in the course that a metric space is *separable* (i.e. has a countable dense subset) if and only if it is *second countable* (i.e. has a countable basis for its topology).

The “if” direction holds for any topological space, and the proof is an immediate consequence of the definitions. The “only if” direction fails, and in the rather exotic setting that there can be a countable topological space with no countable basis for the topology. Take the filter on $\mathbb{N} \times \mathbb{N}$ consisting of sets $A \subset \mathbb{N} \times \mathbb{N}$ such that each $n \in \mathbb{N}$ the vertical strip $A_n = \{m \in \mathbb{N}: (n, m) \in A\}$ has finite complement. Take the topology consisting of the filter along with the empty set. The topology is not even first countable, since given any $(n_0, m_0) \in \mathbb{N} \times \mathbb{N}$ and countable $\mathcal{S}$ of elements of the filter containing $(n_0, m_0)$ we can build an $A$ in the filter containing $(n_0, m_0)$ such that at each $B \in \mathcal{S}$ there will be an $n$ with $A_n$ not including $B_n$. 