Homework Set Two, Math 223A

April 25, 2001

Saturated Models

1. Show that a countable complete theory has a countable saturated model if and only if it has only countably many (consistent) types. \(^1\)

2. Any two countable saturated models with the same theory are isomorphic. \(^2\)

3. Let \(c_q\) be a constant symbol for each \(q \in \mathbb{Q}\). Let \(\mathcal{M} = (\mathbb{Q}, <, (c_q)_{q \in \mathbb{Q}})\) be the expansion of the rationals obtained by interpreting each \(c_q\) to be equal to \(q\). Show that \(\text{Th}(\mathcal{M})\) has \(2^{\aleph_0}\) many types, and hence has no saturated model.

4. But on the other hand, \(\text{Th}(\mathbb{Q}, <)\) (without the resplendent introduction of a cascade of new constant symbols) has a countable saturated model.

5. Does \(\mathcal{M} = \mathbb{N}\) with the usual structure (i.e. \((\omega; +, \times, S, 0, 1)\)) have a countable saturated model? \(^3\)

6. Let \(\mathcal{L}\) be the language generated by unary predicates \((U_n)_{n \in \mathbb{N}}\), and for \(\mathcal{M}\) an \(\mathcal{L}\)-structure and \(a \in \mathcal{M}\) we let \(f_a : \mathbb{N} \to \{0, 1\}\) be the infinite binary sequence given by

\[ f_a(n) = 1 \text{ iff } \mathcal{M} \models U_n(a). \]

Let \(S \subseteq 2^{<\mathbb{N}}\) be a closed downward subset of infinite binary sequences. Let \(T_S\) be the theory whose models are exactly \(\mathcal{M}\) such that

\[ S = \{ f_a | k : k \in \mathbb{N} \}. \]

When does \(T_S\) have a countable saturated model?

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\(^1\) The “only if” direction is pretty clear. As for the converse direction, show that we may build an elementary chain \(\mathcal{M}_0 < \mathcal{M}_1 < \mathcal{M}_2 \ldots\) such that at each \(i\) and for each \(\bar{a} \in \mathcal{M}_i\), we have every type consistent over \((\mathcal{M}_i, \bar{a})\) is realized in \(\mathcal{M}_{i+1}\).

\(^2\) Note! This is exactly the same result as for atomic models, but the reasoning is rather different.

\(^3\) Hint: Think of the Gödel coding for finite subsets of \(\mathbb{N}\).