3. Associated polynomial is \( x^2 + a\lambda + b = 0 \)
   (remember: given \( a > 0, b > 0 \)). Roots are \( \lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \).
   Now if \( a^2 - 4b < 0 \), then the general solution is
   \[ C_1 e^{-a x/2} \cos \left( \frac{x}{2} \sqrt{a^2 + 4b} \right) + C_2 e^{-a x/2} \sin \left( \frac{x}{2} \sqrt{a^2 + 4b} \right). \]
   This comes from \( \lambda_1, \lambda_2 = \frac{-a \pm i \sqrt{-a^2 + 4b}}{2} \) so complex solutions (independent)
   \( e^{\frac{-a x}{2}} \left( e^{\frac{i x}{2} \sqrt{-a^2 + 4b}} \right) \) and \( e^{\frac{-a x}{2}} \left( e^{-\frac{i x}{2} \sqrt{-a^2 + 4b}} \right) \).

   Since \( a > 0 \), these solutions are clearly transient.

   If \( a^2 - 4b \geq 0 \) then \( \sqrt{a^2 - 4b} \) is real and
   \( 0 \leq \sqrt{a^2 - 4b} < a \) (actually \( < a \), since \( b > 0 \)).

   So the two independent solutions are
   \( e^x (-a + \sqrt{a^2 - 4b}) \) and \( e^x (-a - \sqrt{a^2 - 4b}) \) and
   in both the \( x \)-coefficients in the exponents are negative. So these are transient and hence any constant-coefficient linear combination of them (the general solution) is also transient.

   [Think about what happens when \( a \to 0 \): the two roots become purely imaginary and the general solution becomes the non-transient \( C_1 \cos(\sqrt{b} x) + C_2 \sin(\sqrt{b} x) \); this is just putting \( a = 0 \) in the case of \( a^2 - 4b < 0 \).]
1. Rough statement (which is all we have at the moment!):

An $n$th order differential equation
\[
\frac{d^n y}{dx^n} = F(\frac{dy}{dx}, \dots, \frac{d^{n-1} y}{dx^{n-1}}, y, x)
\]
has exactly one solution $y$ with $y(x_0), \dots, y^{(n-1)}(x_0)$ having specified values.

For future reference, here is a more precise statement:

Given $x_0$ and $a_0, \dots, a_{n-1}$, suppose $F$ is continuously differentiable in some neighborhood of the point $(a_0, \dots, a_{n-1})$ in $\mathbb{R}^n$. Then there is an $\varepsilon > 0$ and a function $y$ on $(x_0-\varepsilon, x_0+\varepsilon)$ (interval in $\mathbb{R}$) such that $y$ is $n$-times differentiable on $(x_0-\varepsilon, x_0+\varepsilon)$, $y$ satisfies the differential equation, and

$y(x_0) = a_0$, $y'(x_0) = a_1$, \dots $y^{(n-1)}(x_0) = a_{n-1}$

(this is "existence").

Uniqueness part: If $y_2$ is a function on $(x_0-\varepsilon, x_0+\varepsilon)$ which satisfies the differential equation and which has $y_2(x_0) = a_0$, $y'_2(x_0) = a_1$, \dots $y^{(n-1)}_2(x_0) = a_{n-1}$, then $y_2 = y$ on $(x_0-\varepsilon, x_0+\varepsilon)$. 

5. With the $c_j$’s chosen as in the suggestion, 
the function $F = \sum c_j y_j$ has value 0 at $x_0$, 
first derivative = 0 at $x_0$, \ldots, $(n-1)$st derivative 
= 0 at $x_0$. Since the differential equation is homogeneous 
linear, $F$ is a solution of the equation. So 
is the $0$ function. Since $F$ and the $0$-function 
have the same value and $(n-1)$ derivatives at $x_0$, 
$F = 0$ function by uniqueness. Thus 
$\sum c_j y_j = 0$. This implies that at any 
other point $x_1$, 
$(\sum c_j y_j)(x_1) = \sum c_j y_j(x_1) = 0$ 
$(\sum c_j y_j)'(x_1) = \sum c_j y_j'(x_1)$, \ldots, $(\sum c_j y_j)^{(n-1)}(x_1)$ 
= $\sum c_j y_j^{(n-1)}(x_1) = 0$ so that the column vectors 
\[
\begin{pmatrix}
  y_j(x_1) \\
  y_j'(x_1) \\
  \vdots \\
  y_j^{(n-1)}(x_1)
\end{pmatrix}
\] 
j = 1, \ldots, n
are dependent and hence 
\[
\det\begin{pmatrix}
  y_1(x_1) & \cdots & y_n(x_1) \\
  \vdots & \ddots & \vdots \\
  y_1^{(n-1)}(x_1) & \cdots & y_n^{(n-1)}(x_1)
\end{pmatrix} = 0
\]
6. This follows from the argument used for problem 5 since if all the \( y_j(x_0) \) are 0, then the vectors
\[
\begin{pmatrix}
  y_j(x_0) \\
y_j'(x_0) \\
\vdots \\
y_j^{(n-1)}(x_0)
\end{pmatrix}
= \begin{pmatrix}
  0 \\
y_j'(x_0) \\
\vdots \\
y_j^{(n-1)}(x_0)
\end{pmatrix},
\]
are necessarily dependent (reason: associated determinant clearly = 0 or you can observe that they are \( n \) vectors in the \((n-1)\) dimensional subspace consisting of all vectors with first component = 0).

7. \( k = 1 \):
\[
(\Delta - \alpha) (x e^{\alpha x}) = e^{\alpha x} + x \alpha e^{\alpha x} - \alpha (x e^{\alpha x})
= e^{\alpha x} = k! e^{\alpha x} \text{ for } k = 1.
\]

Assuming works for \( k \geq 1 \), then
\[
(\Delta - \alpha)^{k+1} (x e^{\alpha x}) = (\Delta - \alpha)^k [(\Delta - \alpha) x^{k+1} e^{\alpha x}]
= (\Delta - \alpha)^k \left[(k+1) x^k e^{\alpha x} + x^{k+1} \alpha e^{\alpha x} - \alpha x^{k+1} e^{\alpha x}\right]
= (\Delta - \alpha)^k \left[(k+1) x^k e^{\alpha x}\right] = (k+1) (\Delta - \alpha)^{k+1} (x^k e^{\alpha x})
= (k+1) k! e^{\alpha x} = (k+1)! e^{\alpha x}
\]
\[\square\]

Induction hypothesis
8. If \( P(\lambda) = (\lambda - \alpha)^k \) and \( Q(\lambda) = Q(\lambda)(\lambda - \alpha)^k \), then
\[
P(D) = Q(D)(D - \alpha)^k.
\]
Then
\[
P(D)\left( \frac{1}{k! Q(\alpha)} x^k e^{\alpha x} \right)
\]
\[= \frac{1}{k! Q(\alpha)} Q(D) \left[ (D - \alpha)^k (x^k e^{\alpha x}) \right]
\]
\[= \frac{1}{k! Q(\alpha)} Q(D) \left[ k! e^{\alpha x} \right] = \frac{1}{k! Q(\alpha)} k! Q(\alpha) e^{\alpha x} = e^{\alpha x}
\]

9. Actually, one of the ways problem 2 was solved follows immediately from \( D e^{\alpha x} = \alpha e^{\alpha x} \), every \( \lambda > 0 \).

10. General solution of the differential equation \( \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 0 \):

\[
P(D) = D^2 - D + 1 = (D - 1)^2.
\]

So independent solutions are \( e^x \) and \( x e^x \). So \( y_1 = e^x, y_2 = x e^x \).

\[
\begin{align*}
    y_1' &= e^x, \quad y_2' = e^x + x e^x. \quad \text{Solve} \quad e^x v_1' + x e^x v_2' = 0 \\
    (e^x) v_1' + (e^x + x e^x) v_2' &= e^x
\end{align*}
\]

Taking difference \( v_2' (e^x + x e^x) = e^x \) or \( v_2' = e^x \).

So \( v_1' = -x \). Thus \( v_1 = -\frac{x^2}{2} + C_1, v_2 = x + C_2 \).

Solution \( -\frac{x^2}{2} e^x + x e^x + C_1 e^x + C_2 x e^x = \frac{x^2}{2} e^x + \text{gensol of homogenous solution} \).
Check \( y = \frac{x^2}{2} e^x \)  
\[ y' = \frac{x}{1} e^x + \frac{x^2}{2} e^x \]  
\[ y'' = e^x + xe^x + \frac{x^2}{2} e^x \]  
\[ y'' - 2y' + y = e^x + 2xe^x + \frac{x^2}{2} e^x \]
\[ \quad - 2xe^x + x^2 e^x = e^x \quad \checkmark \]

11. Since determinant = 0 if \( \lambda_i = \lambda_j \quad i \neq j \)

it follows that (as polynomial in \( \lambda_1, \ldots, \lambda_n \))
det is divisible by \( \lambda_i - \lambda_j \) all \( i,j \) with \( i \neq j \).
So det is divisible by \( \prod_{i \neq j} (\lambda_i - \lambda_j) \).

Now det has total degree \( 1 + 2 + \ldots + n - 1 = n(n-1)/2 \);
and so does \( \prod_{i \neq j} (\lambda_i - \lambda_j) \) — factors are linear
and there are \( \frac{n(n-1)}{2} \) of them. So

\( \prod_{i \neq j} (\lambda_i - \lambda_j) \) divides det \( \Rightarrow \) det = (cons.) \( \prod_{i \neq j} (\lambda_i - \lambda_j) \)

Checking coefficient of \( \lambda_1^{n-1} \lambda_2^{n-2} \ldots \lambda_n \) in both
(coefficient is 1 in both cases) gives cons. = 1.

Hence det = 0 \( \iff \) some \( \lambda_i = \text{some} \lambda_j \quad i \neq j \).

And if \( \lambda_i \)’s are all distinct, then det \( \neq 0 \).
12. (Done in notes) If \( C_1 e^{\lambda_1 x} + \ldots + C_n e^{\lambda_n x} = 0 \)
then (supposing without loss of generality that \( \lambda_1 < \lambda_2 < \ldots < \lambda_n \)) multiplying by \( e^{-\lambda_n x} \) gives
\( C_1 e^{(\lambda_1 - \lambda_n) x} + \ldots + C_{n-1} e^{(\lambda_{n-1} - \lambda_n) x} + C_n = 0 \). Let
\( x \to +\infty \) to get \( C_n = 0 \). Continue to get all \( C_i = 0 \).

13. By prob. 12, \( e^{\lambda_1 x}, \ldots, e^{\lambda_n x} \) are independent solutions.

\[
\det \begin{pmatrix} y_1(0) & y_n(0) \\ y_1'(0) & y_n'(0) \\ \vdots & \vdots \\ y_1^{(n-1)}(0) & y_n^{(n-1)}(0) \end{pmatrix} = \det \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \lambda_1^2 & \ldots & \lambda_n^2 \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix}
\]

with \( y_j = e^{\lambda_j x} \). Now problem 11 verifies the nonzerosness of the determinant, which confirms the independence of the solutions. Problem 5 then implies that

\[
\det \begin{pmatrix} y_1(x) & y_n(x) \\ y_1'(x) & y_n'(x) \\ \vdots & \vdots \\ y_1^{(n-1)}(x) & y_n^{(n-1)}(x) \end{pmatrix}
\]
should be nonzero for every \( x \).

This
\[
\begin{pmatrix} e^{\lambda_1 x} & \ldots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \ldots & \lambda_n e^{\lambda_n x} \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \ldots & \lambda_n^{n-1} e^{\lambda_n x} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 x} & \ldots & e^{\lambda_n x} \\ \lambda_1 & \ldots & \lambda_n \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix}
\]
so this works. And this latter calculation shows that it really is true that nonzero at one point implies nonzero at every point since the values are all nonzero (exponential) multiples of the value at 0. [In fact if we write $W/(e^{x_1} \cdot \cdot \cdot e^{x_n})$ for the determinant, then

$$W = (\text{cons.}) e^{(\lambda_1 + \cdot \cdot \cdot + \lambda_n)x}$$

So

$$W' = \prod_{j=1}^{n} \lambda_j W$$

where

$$\sum \lambda_j =$$

- the $x^{n-1}$ coefficient of $P(x)$ or $W$ satisfies

$$W' = -p_1 W$$

if the differential equation is

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdot \cdot \cdot p_n y = 0.$$ 

This turns out to work even when the $p_i$ are functions, not constants: i.e., $W' = -p_1 W$ in the general case, even though the form of the solutions $y$ as exponentials may not hold in this more general situation?

Note: Remarks in the bracket [ ] will not be on the midterm.