Solutions of Sample Problems: Part I

1. (1) \( \gamma' - \gamma = 0 \), \( \gamma'' - \gamma = 0 \). \( P(\lambda) = \lambda - 1 \).
Solution: \( \gamma = C_1 e^x \), \( C \) constant. These are only solutions since \( C \) = given \( \gamma(0) \) value solves initial value problem (at \( x = 0 \)), hence gives general solution by existence and uniqueness theorem. Ans. \( \gamma = C_1 e^x \).

2. \( \gamma'' - \gamma = 0 \). \( P(\lambda) = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1) \).
Solutions \( e^x, e^{-x} \): general solution \( C_1 e^x + C_2 e^{-x} \).
These are only solutions because \( \gamma = C_1 e^x + C_2 e^{-x} \)
has \( \gamma(0) = C_1 + C_2 \), \( \gamma'(0) = C_1 - C_2 \) so \( \gamma(0), \gamma'(0) \)
can be arbitrarily specified: \( C_1 + C_2 = A \), \( C_1 - C_2 = B \) is always solvable for \( C_1, C_2 \) given \( A, B \) because

\[
\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0.
\]
Ans. \( \gamma = C_1 e^x + C_2 e^{-x} \).

3. \( \gamma''' - \gamma = 0 \). \( P(\lambda) = \lambda^3 - 1 \).
\( \lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1) \), roots \( \lambda = 1, \frac{-1 + \sqrt{3}i}{2}, \frac{-1 - \sqrt{3}i}{2} \).

Associated solutions \( e^x, e^{-x/2} e^{(\sqrt{3}/2)x}, e^{-x/2} e^{-i(\sqrt{3}/2)x} \),
or for real roots \( e^x, e^{-x/2} \cos \left( \frac{\sqrt{3}}{2} x \right), e^{-x/2} \sin \left( \frac{\sqrt{3}}{2} x \right) \).
These are independent; either use general theorem on independence or note that \( \begin{pmatrix} \gamma_1, \gamma_2, \gamma_3 \\ \gamma_1', \gamma_2', \gamma_3' \\ \gamma_1'', \gamma_2'', \gamma_3'' \end{pmatrix} \) matrix

at 0 is nonsingular (det \( \neq 0 \)). Second way is a little messy, good time to use independence argument:

\( C_1 e^x + C_2 e^{-x/2} \cos \left( \frac{\sqrt{3}}{2} x \right) + C_3 e^{-x/2} \sin \left( \frac{\sqrt{3}}{2} x \right) \equiv 0 \implies C_1 = 0 \) (let \( x \to +oo \)) so \( e^{-x/2} \left( C_2 \cos \left( \frac{\sqrt{3}}{2} x \right) + C_3 \sin \left( \frac{\sqrt{3}}{2} x \right) \right) \equiv 0 \).
$\Rightarrow \quad C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_3 \sin\left(\frac{\sqrt{5}}{2}x\right) = 0 \quad \Downarrow \quad C_2 = 0, \; C_3 = 0.$

(For midterm you could just quote theorem. I summarized how the proof of the independence theorem works in this case. You could just quote the independence theorem, that if all $\lambda$'s are distinct, then $e^{\lambda x}$'s are independent.)

2. $\frac{d^2y}{dx^2} + k^2 y = \sin bx$. To solve this, first do

$$\frac{d^2y}{dx^2} + k^2 y = e^{ibx} \quad \text{or} \quad (D^2 + k^2)y = e^{ibx}.$$  

Solutions:

$$P(\lambda) = \lambda^2 + k^2 = (\lambda + ik)(\lambda - ik)$$

(1) If $(ib)^2 + k^2 \neq 0$ (same as $P(ib) \neq 0$) roots are $+ib$.

then $\frac{1}{P(ib)} e^{ibx}$ is solution, $\frac{1}{P(ib)} e^{ibx} = \frac{1}{k^2 - b^2} e^{ibx}$

Taking imaginary part gives $\frac{1}{k^2 - b^2} \sin(bx)$ solves

$$\frac{d^2y}{dx^2} + k^2 y = \sin bx.$$  General solution:

$$= \frac{1}{k^2 - b^2} \sin(bx) + C_1 \cos(kx) + C_2 \sin(kx)$$

since general solution of $\frac{d^2y}{dx^2} + k^2 y = 0$ is

$$C_1 \cos(kx) + C_2 \sin(kx).$$

(2) If $(ib)^2 + k^2 = 0$ then $P(\lambda) = (\lambda - ib)(\lambda + ib)$

since $P(\lambda) = \lambda^2 + k^2$ and $k = b$. Solution is then obtained (for $\frac{d^2y}{dx^2} + k^2 y = e^{ibx}$) by noting $Q(\lambda) = \lambda^2$
In our usual notation, \( P(\lambda) = (\lambda - ib) Q(\lambda), \quad Q(ib) \neq 0 \)

so solution is \( \frac{1}{Q(ib)} x e^{ibx} = \frac{1}{2ib} x e^{ibx} \)

To get solution of \( \frac{d^2y}{dx^2} + k^2 y = \sin b x \), take imaginary part of \( \frac{1}{2ib} x e^{ibx} = \frac{1}{2ib} x (\cos bx + i \sin bx) \)

which has imaginary part \( = -\frac{1}{2b} x \cos bx \).

Does this work? If \( y = -\frac{1}{2b} x \cos bx \) then

\[ y' = -\frac{1}{2b} \cos bx + \frac{b}{2b} x \sin bx \quad \text{and} \]

\[ y'' = +\frac{1}{2} \sin bx + \frac{i}{2} \sin bx + \frac{b}{2} x \cos bx \quad \text{so simplified} \]

since \( k^2 = b^2 \):

\[ y'' + k^2 y = \sin bx + \frac{b}{2} x \cos bx + b^2 \left( -\frac{1}{2b} x \cos bx \right) \]

\[ = \sin bx \]

as required. Note how tricky this is! The \( Q \)-method really amounts to something! It is much easier!

Alternative method for case 2: \( (D^2 + b^2)y = e^{ibx} \)

so \( (D + ib)[D - ib)y] = e^{ibx} \)

so setting \( G = (D - ib)y, \quad (D + ib)G = e^{ibx} \)

and hence

\[ D(e^{ibx} G) = e^{2ibx} \quad \text{or} \quad e^{ibx} G = \frac{1}{2ib} e^{2ibx} + C \quad \text{or} \]

\[ G = \frac{1}{2ib} e^{ibx} + C e^{-ibx}. \]

Then \( (D - ib)y = \frac{1}{2ib} e^{ibx} + C e^{-ibx} \)

\[ + e^{ibx} \]

\[ = \frac{1}{2ib} e^{ibx} + C e^{-ibx} \]
\[ D(e^{-ibx}y) = \frac{1}{2ib} + Ce^{-2ibx} \]

or

\[ e^{-ibx}y = \frac{1}{2ib}x + C \frac{e^{-2ibx}}{-2ibx} + C_2 \]

or

\[ y = \frac{1}{2ib}xe^{ibx} + \frac{C}{-2ibx}e^{-ibx} + C_2e^{ibx} \]

general solution of
homogeneous equation

And again, imaginary parts gives

\[ y = -\frac{1}{2b}x \cos bx + \text{general real solution of} \]

homogeneous equation.

Notice that this second method is much longer and trickier than the P,Q method used to begin with! It did work, but it was messy.

Physical interpretation of \( b=k \) case: The equation described an undamped mass/spring oscillator which was being driven at its "natural resonance" = the way it oscillates when set in motion but then no force applied. Driving the oscillator at this frequency results in larger and larger amplitudes: \( -\frac{1}{2b}x \cos bx \) as \( x \rightarrow +\infty \) is unbounded.