Sample Problems for Midterm (continued)

14. Solution (particular) is \( \frac{1}{p(iw)} e^{iwx} \) for \( \text{RHS} = e^{iwx} \)

where \( p(\lambda) = \lambda^2 + \pi \lambda + k \). (Note that \( p(iw) \)

\( \neq 0 \) since roots of \( p(\lambda) \) are \( \frac{-\pi \pm \sqrt{\pi^2 - 4k}}{2} \),

which cannot be pure imaginary. Another reason: \( p(iw) \)

\( = -w^2 + k + i2w \) which cannot be 0 since \( \text{Im part} = 2w \)

and \( p(0) \neq 0 \) to \( w \neq 0 \) and hence \( 2w \neq 0 \) is only possibility.

Now:

\[ \frac{1}{k - w^2 + i2w} (\cos wx + i \sin wx) = \frac{1}{p(iw)} e^{iwx} \]

So:

\[ \frac{iwx}{p(iw)} = \frac{(k - w^2) - i2w}{(k - w^2)^2 + (2w)^2} (\cos wx + i \sin wx). \]

Taking imaginary parts:

\[ \text{Im} \left[ \frac{iwx}{p(iw)} \right] = \frac{1}{(k - w^2)^2 + (2w)^2} \left[ (k - w^2) \sin wx - (2w) \cos wx \right] \]

There is an angle \( \phi \) with \( \cos \phi = \frac{k - w^2}{\sqrt{(k - w^2)^2 + (2w)^2}} \) and \( \sin \phi = \frac{2w}{\sqrt{(k - w^2)^2 + (2w)^2}} \),

because the sum of squares of these two things is 1. Then

\[ \text{Im} \left[ \frac{iwx}{p(iw)} \right] = \sin \left( wx - \phi \right) \]

by usual formula for \( \sin(A - B) \). So particular solution of diff eq, \( \text{RHS} = \sin wx \)

is:

\[ \frac{1}{\sqrt{(k - w^2)^2 + (2w)^2}} \sin \left( wx - \phi \right). \]

General sol = this + general solution to homogeneous equation, which (as was shown earlier) gen hom. sol is transient.
This finishes part (a). For part (b), note that max amplitude of the \( \sqrt{\frac{1}{\sin^2(\omega t - \phi)}} \) part occurs for that \( \omega \) value (if any) for which \( \frac{1}{\sin^2} \) is minimum, i.e. \( (k - \omega^2)^2 + T^2 \omega^2 \) attains its minimum. This is \( \omega^4 + \left( \frac{T^2 - 2k}{2} \right) \omega^2 + \frac{k^2}{2} \). Think of this as a quadratic polynomial in \( \omega^2 \geq 0 \). Its minimum point on \([0, \infty)\) is either \( \omega = 0 \) or where \( \omega^2 = -\text{coefficient of } \omega^2 = \frac{2k - T^2}{2} \)

So condition for a "nondegenerate" minimum \( (\omega^2_{\text{min}} > 0) \) is 
\[-T^2 + 2k > 0. \]
Assuming this condition is met, minimum is
\[
(\omega^2 - \frac{T^2}{2})^2 + \left( \frac{1}{2}(T^2 - 2k)(2k - T^2) + k^2 \right)
= k^2 - \frac{T^2}{2} k + \frac{T^4}{4} + \frac{1}{2}(-T^4 + 4k^2 + 4kT^2) + k^2
= -\frac{1}{4} T^4 - k^2 + kT^2
= (\frac{1}{4})(-T^4 + 4kT^2) = \frac{1}{4} T^2 (4k - T^2)
\]
Note that \( 4k - T^2 > 0 \) since \( 2k - T^2 > 0 \). So max amplitude is
\[
\frac{1}{\sqrt{\frac{1}{4} T^2 (4k - T^2)}} = \frac{\sqrt{4}}{\sqrt{T^2 (4k - T^2)}} = \frac{1}{T (k - \frac{T^2}{2})}.
\]
Note that when \( T \) is small (not much damping), this is large. Also when \( \omega \) is small, max amplitude occurs near \( \omega^2 = k \), the undamped resonance.