Constant Coefficient Linear Differential Equations: Lecture II

Equations of the form
\[ \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + p_n y = F(x) \]  \hspace{1cm} (\star)

where \( p_1, \ldots, p_n \) are constants (numbers).

Later we shall have a procedure for solving for \( y \), given \( n \) "independent" solutions \( y_1, \ldots, y_n \) of the equation, the associated "homogeneous" equation
\[ \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + p_n y = 0 \]  \hspace{1cm} (\star\star)

(We shall define "independent" in a moment. The procedure from going from the general solution of the homogeneous equation to the solution of the equation when \( F \neq 0 \) actually will work when the \( p_1, \ldots, p_n \) are functions, not necessarily constants. This will be covered later).

Notation: Write \( D = \frac{d}{dx} \) and the homogeneous equation (\star\star) as
\[ D^n y + p_1 D^{n-1} y + \ldots + p_{n-1} Dy + p_n y = 0. \]

We associate to this a polynomial \( P(\lambda) \) defined by
\[ P(\lambda) = \lambda^n + p_1 \lambda^{n-1} + \ldots + p_{n-1} \lambda + p_n. \]
The polynomial $P(\lambda)$ can be factored

\[ P(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \]

where the $\lambda_i$'s are numbers, but they may need to be complex numbers, e.g.

\[ D^2 + 1, \quad \lambda^2 + 1 = (\lambda - i)(\lambda + i) \]

Equation \[ \frac{d^2 y}{dx^2} + y = 0 \]

The factorization of $P(\lambda)$ corresponds to a factorization of the left-hand side of the equation:

\[ \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y \]

\[ = \left( \frac{d}{dx} - \lambda_1 \right) \left[ \left( \frac{d}{dx} - \lambda_2 \right) \left[ \cdots \left( \frac{d}{dx} - \lambda_n \right) y \right] \right] \]

\[ = (D - \lambda_1)(D - \lambda_2)\cdots(D - \lambda_n)y \]

Written as

\[ (D - \lambda_1)(D - \lambda_2)\cdots(D - \lambda_n)y \]

Example \[ \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + y \]

\[ = \left( \frac{d}{dx} - 3 \right) \left[ \left( \frac{d}{dx} - 2 \right) y \right] \]

\[ = (D - 3)(D - 2)y. \]
This makes sense because, once all coefficients are constant,

\[
(\mathcal{D} - 3) \left[ (\mathcal{D} - 2) y \right] = \left( \frac{d}{dx} - 3 \right) \left[ \frac{d^2 y}{dx^2} - 2 y \right]
\]

\[
= \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} - 3 \left( \frac{dy}{dx} - 2 \right) = \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6 y.
\]

The \((\mathcal{D} - \lambda i)\) factors commute as "operators" just the way the \(\lambda - \lambda i\) factors commute in writing the polynomial \(P(\lambda)\) as a product.

Why is this useful? The reason is that we already know how to solve \((\mathcal{D} - \lambda i) G = 0\) (or \(P(\epsilon)\)) given a number \(\lambda\), complex or not. So we can solve \((\mathcal{D} - \lambda i) G = 0\) for that matter \((\mathcal{D} - \lambda i) G = 0\) by solving first order linear equations successively.

Example: Find the general solution of

\[
\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6 y = 0.
\]

Answer: \((\mathcal{D}^2 - 5 \mathcal{D} + 6) y = (\mathcal{D} - 3) (\mathcal{D} - 2) y\).

Let \(G(x) = (\mathcal{D} - 2) y\). Then \((\mathcal{D} - 3) G = 0\)

So (as in Lecture I)

\[ G(x) = Ae^{3x} \quad \text{for some constant } A \]

Then \((\mathcal{D} - 2) y = G = Ae^{3x}\). Solving (using "integrating factor" \(e^{-2x}\)) gives

\[
\mathcal{D} (e^{-2x} y) = Ae^x \quad \text{or} \quad e^{-2x} y = Ae^x + B
\]

for \(B\) another constant or \(y = Ae^{3x} + Be^{2x}\).\]
If we did this in the opposite order \((D-2)(D-3)y=0\):
\[G = (D-3)y \quad \text{so} \quad G(x) = Ae^{2x}\]
and
\[D(e^{-3x} G) = A e^{2x} e^{-3x} = Ae^{-x}\]
so
\[e^{-3x} y = B + \int Ae^{-x} = B - Ae^{-x}\]
and
\[y = Be^{3x} - Ae^{2x}\]
The answer looks different. For one thing, there is a minus sign. But since \(A\) and \(B\) are arbitrary constants, the sets of all solutions you get are really the same in both cases.

Clearly, this process works for the general 2nd order case and the \(n>2\) cases, too. One just peels off the layers \((D-\lambda i)\)'s by solving successive first-order linear equations.

What you end up with in general looks though it might be complicated. But in fact, for the homogeneous case \((**\)) at least, it is relatively simple to describe:

1) If the roots \(\lambda_1, \ldots, \lambda_n\) of \(P(\lambda)=0\) \((i.e.\), \(P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)\) are all real and different from each other, then the general solution of \(P(D)y=0\) is
\[C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \ldots + C_n e^{\lambda_n x}\]
the \(C_i\)’s numbers. We say the general solution is a linear combination of \(e^{\lambda_1 x}, \ldots, e^{\lambda_n x}\).
(2) If some of the \( \lambda_i \)'s in the factorization
\[ P(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \]
are the same, then each group of, say, \( k \) \( \lambda_i \)'s all equal

(last different from the rest of the \( \lambda_i \)'s) gives

"particular solutions"

\[ e^{\lambda_i x}, \ x e^{\lambda_i x}, \ldots, x^k e^{\lambda_i x} \]

And the general solution is the set of all
linear combinations of these.

Example: (a) \( \frac{d^2}{dx^2} - 4 \frac{dy}{dx} + y = 0 \)
\( (D - 2)^2 y = 0 \).
General sol: \( C_1 e^{2x} + C_2 e^{-2x} \)

(b) \( (D - i)^2(D + i)^2 y = 0 \) or \( \frac{d^4 y}{dx^4} + 2 \frac{d^2 y}{dx^2} + y = 0 \)
General solution
\[ C_1 e^{ix} + C_2 e^{ix} + C_3 e^{-ix} + C_4 e^{-ix} \]
In real terms, this becomes for the general real solution

\[ A_1 \cos x + A_2 \sin x + A_3 x \cos x + A_4 x \sin x \]
There are four (independent) \( \text{real} \) constants, corresponding
to the equation \( \frac{d^4 y}{dx^4} + 2 \frac{d^2 y}{dx^2} + y = 0 \) being

of degree 4. This real form comes from
writing \( e^{ix} = \cos x + i \sin x \) and \( e^{-ix} = \cos x - i \sin x \)
and collecting terms and then seeing
what is needed for the result to be real-valued.
The occurrence of \( n \) real constants in getting the general solution of the \( n \)th order differential (homogeneous) equation is guaranteed by the following:

The set of all solutions of

\[
\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + p_n y = 0
\]

is a vector space: solutions can be added and multiplied by constants to give other solutions. The general theorem on existence and uniqueness says that this vector space \( S \) (for solutions) is mapped one-to-one (uniqueness) and onto (existence) \( \mathbb{R}^n \) by the linear transformation

\[
y \rightarrow (y(0), \frac{dy}{dx}(0), \ldots, \frac{d^{n-1} y}{dx^{n-1}}(0))
\]

\( y \in S \). So the vector space \( S \) is dimension exactly \( n \).

[Note: This reasoning does not depend on the \( p_1 \)s being constant: it works for linear homogeneous equations in general].
It is interesting to watch this general idea in action in concrete cases. Consider, for example, the case where the \( p_i \)'s are constants and all the \( \lambda_i \)'s are distinct and real. In this case, the space \( S \) is supposed to consist of all functions of the form

\[
y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \ldots + C_n e^{\lambda_n x}
\]

\( C_i \)'s numbers. The linear transformation

\[
y \rightarrow \begin{pmatrix} y(0), \frac{dy}{dx}(0), \ldots, \frac{d^{n-1}y}{dx^{n-1}}(0) \end{pmatrix}
\]

sends \( C_1 e^{\lambda_1 x} + \ldots + C_n e^{\lambda_n x} \) to

\[
(C_1 + C_2 + \ldots + C_n, \lambda_1 C_1 + \ldots + \lambda_n C_n, \lambda_1^2 C_1 + \ldots + \lambda_n^2 C_n, \ldots, \lambda_1^{n-1} C_1 + \ldots + \lambda_n^{n-1} C_n)
\]

This transformation is supposed to be 1-1; onto. So the system of equations, \( C_i \)'s regarded as unknowns, \( A_0, \ldots, A_{n-1} \) arbitrarily given

\[
C_1 + C_2 + \ldots + C_n = A_0
\]

\[
\lambda_1 C_1 + \lambda_2 C_2 + \ldots + \lambda_n C_n = A_1
\]

\[
\lambda_1^{n-1} C_1 + \lambda_2^{n-1} C_2 + \ldots + \lambda_n^{n-1} C_n = A_{n-1}
\]

ought to have one and only one solution for each given set of \( A_0, \ldots, A_{n-1} \). This will be true (by linear algebra) if the determinant of the coefficients

\[
\begin{vmatrix}
\lambda_1 & 1 & \ldots & 1 \\
\lambda_1^{n-1} & \lambda_2 & \ldots & \lambda_n \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1}
\end{vmatrix} \neq 0.
\]
This determinant is the famous
van der Monde determinant, which is
well known to be nonzero exactly when
the $\lambda_1, \ldots, \lambda_n$ are all distinct.

In fact

$$\det \left( \begin{array}{ccc}
\lambda_1 & \cdots & \lambda_n \\
\lambda_1^2 & \cdots & \lambda_n^2 \\
\vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \cdots & \lambda_n^{n-1}
\end{array} \right) = \text{the product of all } \lambda_j - \lambda_i, \ j > 1.$$

Examples: $\det \left( \begin{array}{cc}
\lambda_1 & \lambda_2 \\
\lambda_1 & \lambda_2
\end{array} \right) = \lambda_2 - \lambda_1$

$$\det \left( \begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2
\end{array} \right) = (\lambda_1, \lambda_3^2 - \lambda_3 \lambda_2^2)
- \lambda_1 (\lambda_3^2 - \lambda_2^2)
+ \lambda_1^2 (\lambda_3 - \lambda_2)
= (\lambda_3 - \lambda_2) \left[ \lambda_3 \lambda_3 - \lambda_1 (\lambda_3 + \lambda_2) + \lambda_1^2 \right]
= (\lambda_3 - \lambda_2) \left[ -\lambda_1 (\lambda_3 - \lambda_1) + \lambda_2 (\lambda_3 - \lambda_1) \right]
= (\lambda_3 - \lambda_2) (\lambda_3 - \lambda_1) (\lambda_2 - \lambda_1).$$

The proof in general of the $\det = \text{product formula}$
comes from observing that, since $\det = 0$ if $\lambda_i = \lambda_j, \ j > i$, the $\det$ as a polynomial in $\lambda_1, \ldots, \lambda_n$
must be divisible by $\lambda_j - \lambda_i, \ j > i$. Hence $\det$
must be divisible by $\prod \text{prod}$, and counting degree =
constant multiple of $\text{prod}$. Constant is easily checked to be $+1$. 
Namely, the term of the form

\[ \lambda_{n-1} \lambda_{n-2} \ldots \lambda_2 \]

occurs only once in the determinant expansion (as the main diagonal) and up occurs clearly with coefficient +1.

In the product, this term also appears only once, by choosing \( \lambda_i \) from all \( \lambda_j - \lambda_i \) terms with \( i < j \), and \( j = n \), choosing \( \lambda_{n-1} \) from all \( \lambda_j - \lambda_i \) terms with \( i < j \) and \( j = n-1 \), and so on.

Thus the \( \lambda_{n-1} \ldots \lambda_2 \) term also appears with coefficient +1. So

\[
\det \left( \begin{array}{c|c|c|c}
\vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_n \\
\end{array} \right) = \text{product} \left( \lambda_j - \lambda_i \right) \quad \text{all } i, j \text{ with } j > i
\]

[Noting that both \( \det \) and product are antisymmetric under interchange of a pair of \( \lambda \)'s one can prove the whole formula this way without appealing to the differentiability business at the end of the previous page, if desired].