Solving \[ \frac{d^k y}{dx^k} + p_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + p_1 y = f(x) \]

when \( p_i \)s are constants and \( f(x) = e^{ax} \)

[Here \( a \) can be a complex number!]

With \( P(\lambda) = \lambda^n + p_1 \lambda^{n-1} + \ldots + p_n \) (polynomial of degree \( n \))

Case 1) If \( P(a) \neq 0 \), then \( y = \frac{1}{P(a)} e^{ax} \) solves the equation.

Proof is by direct substitution: \( P(D) e^{ax} = P(a) e^{ax} \).

Case 2) If \( P(a) = 0 \), then let \( k > 0 \) be the largest \( k \) \([\text{largest power of} (\lambda - a)]\) such that \( (\lambda - a)^k \) divides into \( P(\lambda) \), so \( P(\lambda) = (\lambda - a)^k Q(\lambda) \) where \( Q(a) \neq 0 \). Then

\[ y = \frac{1}{k! Q(a)} x^k e^{ax} \] solves the equation.

Proof: \( P(D) = Q(D) (D-a)^k \). Now

\( (D-a)^k (x^k e^{ax}) = k! e^{ax} \) (proof by induction: later).

So \( Q(D) [(D-a)^k x^k e^{ax}] = k! Q(a) e^{ax} \)

since \( Q(D) e^{ax} = Q(a) e^{ax} \) as in case 1).

Induction proof that \( (D-a)^k (x^k e^{ax}) = k! e^{ax} \) for \( k \geq 0 \)

\( k = 1 \) case is direct calculation \( (D-a)(x e^{ax}) = e^{ax} + a x e^{ax} - a x e^{ax} = e^{ax} \).

Induction step: \( (D-a)^{k+1} (x^{k+1} e^{ax}) = (D-a)^k \left[ (D-a) (x^{k+1} e^{ax}) \right] \)

\[ = (D-a)^k \left[ (k+1) x^k e^{ax} + x^{k+1} a x e^{ax} - a x^{k+1} e^{ax} \right] \]

\[ = (k+1) [(D-a)^k (x^k e^{ax})] = (k+1) k! e^{ax} = (k+1)! e^{ax} \]

assuming \( k \) case works, \( k \geq 1 \)