Math 131a Handout #3

We will assume two fundamental properties of \( \mathbb{N} \):

(N1) Every non-empty subset \( S \) of \( \mathbb{N} \) has a least element.

(N2) The Fundamental Theorem of Arithmetic: every number \( n \in \mathbb{N} \) has a unique factorization

\[
n = 2^{a_1}3^{a_2} \cdots
\]

where \( 0 \leq a_k \in \mathbb{N} \cup \{0\} \).

**Theorem 0.1.** Suppose that \( S \) is an infinite subset of a countable set \( T \). Then \( S \) is countably infinite (i.e., \( S \approx \mathbb{N} \)).

**Proof.** First assume that \( T = \mathbb{N} \). We define a function \( f : \mathbb{N} \to T \) by induction. From (N1) we may let \( f(1) = \min S \). Let us suppose that we have defined \( f(n-1) \) (where \( n > 1 \)). Since \( S \) is assumed infinite, \( S \setminus \{f(1), \ldots, f(n-1)\} \) is non-empty, and we may use (N1) to define

\[
f(n) = \min S \setminus \{f(1), \ldots, f(n-1)\}.
\]

It is evident that

\[
f(1) < f(2) < \ldots
\]

and in particular \( f \) is 1-1.

**To see that \( f \) is onto** we have to show that if \( p \in S \), then there is an \( n \) such that \( f(n) = p \). First observe that for all \( n \in \mathbb{N}, n \leq f(n) \). To see this note that \( 1 \leq f(1) \) since 1 is the least element in all of \( \mathbb{N} \). Suppose that we know that \( n \leq f(n) \). Then \( n \leq f(n) < f(n+1) \) implies that \( n+1 \leq f(n+1) \) (note that \( f(n+1) \) is a “whole” number). Thus induction gives the general result \( \forall n, n \leq f(n) \).

Given \( p \in S \), let \( A = \{n \in \mathbb{N} : p \leq f(n)\} \). This is non-empty since \( p \leq f(p) \). Let \( n_0 = \min A \). If \( n_0 = 1 \), then

\[
f(1) = \min S \leq p.
\]

and thus \( f(1) = p \). If \( n_0 > 1 \), then

\[
f(1) < \ldots < f(n_0 - 1) < p \leq f(n_0),
\]

and thus \( p \) is in \( S \setminus \{f(1), \ldots, f(n_0 - 1)\} \). It follows that

\[
f(n_0) = \min S \setminus \{f(1), \ldots, f(n_0 - 1)\} \leq p,
\]

and thus \( f(n_0) = p \).

For the general case, by assumption \( T \approx \mathbb{N} \), i.e., there is a bijection \( g : T \to \mathbb{N} \). Then \( g(S) \) is an infinite subset of \( \mathbb{N} \), and by our previous argument \( g(S) \approx \mathbb{N} \). Since \( S \approx g(S) \), \( S \approx \mathbb{N} \), i.e., \( S \) is countably infinite. \( \square \)

**Theorem 0.2.** Suppose that \( T \) is a countable set and \( f : T \to U \) is onto. Then \( U \) is countable.

**Proof.** Since \( f \) is onto, we have that for each \( u \in U \), the set \( T_u = \{t : f(t) = u\} \) is non-empty. For each \( u \in U \), we choose an element \( t_u \in T_u \). We define \( g : U \to T \) by \( g(u) = t_u \). From this definition \( f(g(u)) = u \). It follows that \( g : U \to T \) is one-to-one since if \( g(u_1) = g(u_2) \), then \( f(g(u_1)) = f(g(u_2)) \) and thus \( u_1 = u_2 \). It is evident that \( g \) is a one-to-one correspondence of \( U \) onto the set \( g(U) \), i.e., \( U \approx g(U) \subseteq T \). Since \( g(U) \) is infinite, we conclude from the previous result that \( g(U) \approx \mathbb{N} \), and thus \( U \approx \mathbb{N} \). \( \square \)
Theorem 0.3 (The principle of induction). Suppose that one has a series of statements $P(1), P(2), \ldots$. Then if $P(1)$ is true, and $P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$, then $P(n)$ is true for all $n$.

Proof. Let us suppose that this is false. Then there exists an $n \in \mathbb{N}$ such that $P(n)$ is false. Thus the set $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$ is non-empty. Using (N1), we may let $n_0 = \min S$. Since $P(1)$ is assumed true, $n_0 > 1$. From the definition of $n_0$, $P(n_0 - 1)$ is true, and $P(n_0)$ is false, contradicting the fact that for all $n$, $P(n) \Rightarrow P(n+1)$.

*This illustrates the law of logic $\sim (Q \Rightarrow R) \iff [Q \text{ and } \sim R]$.

** This illustrates the law of logic $\sim (\forall x \in X)P(x) \iff [(\exists x \in X) \sim P(x)]$.

Completeness axiom for $\mathbb{R}$: Any set which is bounded above has a least upper bound.

Using letters: if you have a subset $S \subseteq \mathbb{R}$ such that $S \leq b$ for some $b \in \mathbb{R}$ (i.e., $s \leq b$ for all $s \in S$), then $S$ has a least upper bound $b_0$ (i.e., $S \leq b_0$ and if $S \leq b$ then $b_0 \leq b$).

Theorem 0.4. $\mathbb{N}$ does not have an upper bound.

Proof. Suppose that $\mathbb{N}$ has an upper bound. Then using the completeness principle, we may let $b_0 = \sup \mathbb{N}$ be the least upper bound for $\mathbb{N}$. We have that $b_0 - 1 < b_0$ implies that $b_0 - 1$ is not an upper bound for $\mathbb{N}$ i.e., $\mathbb{N} \nsubseteq b_0 - 1$ and there is an $n \in \mathbb{N}$ with $b_0 - 1 < n$. But then $b_0 < n + 1 \in \mathbb{N}$, contradicting the fact that $b_0$ is an upper bound for $S$. QED

Corollary 0.5. For any $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Proof. Since $\mathbb{N}$ is not bounded above, there is an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. It follows that $\frac{1}{n} < \varepsilon$. QED

Corollary 0.6. If $a > 0$ and $b > 0$, there is an $n \in \mathbb{N}$ such that $na > b$.

Proof. You prove this!

Assignment 3

1. p. 54: 1, 2
2. Given complete proofs that
   a) $\lim_{n \to \infty} \frac{n}{n+1} = 1$
   b) $\lim_{n \to \infty} \sqrt{n+1} - \sqrt{n} = 0$
   c) $\lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\pi} = 1$ (use the sandwich principle and a geometrical picture)
3. What can be said if $a_n$ is a convergent sequence in $\mathbb{N}$?
4. Consider the set
   \[
   \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \ldots
   \]
   For which numbers $a$ is there a subsequence converging to $a$?
5. a) Show that if $0 < a < 2$, then $a < \sqrt{2a} < 2$. 

b) Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges.

c) Find the limit of the sequence in b).

6. p. 51: 7, 11
7. p. 54: 5, 6.