1. Problem 1.1.1: Find the quotient and remainder when \(a\) is divided by \(b\):

\(a\) \(=\) 302, \(b\) \(=\) 19
\(a\) \(=\) \(-302\), \(b\) \(=\) 19
\(a\) \(=\) 0, \(b\) \(=\) 19

**Answer:**

(a) \(302 = 15 \cdot 19 + 17\), so \(q = 15\) and \(r = 17\)

(b) \(-302 = -16 \cdot 19 + 2\), so \(q = -16\) and \(r = 2\). Note that \(q \neq -16\) and \(r \neq -17\) because the division algorithm states that the remainder, \(r\) must be less than \(b\) and nonnegative.

(c) \(0 = 0 \cdot 19 + 0\), so \(q = 0\) and \(r = 0\).

2. Problem 1.2.1 Find the greatest common divisors

(a) \((56, 72) = ?\)

(b) \((24, 138) = ?\)

(c) \((143, 227) = ?\)

**Answer:**

(a)

\[
\begin{align*}
72 & = 1 \cdot 56 + 16 \\
56 & = 3 \cdot 16 + 8 \\
16 & = 2 \cdot 8 + 0
\end{align*}
\]

Thus \((56, 72) = 8\) by the Euclidean algorithm.

(b)

\[
\begin{align*}
138 & = 5 \cdot 24 + 18 \\
24 & = 1 \cdot 18 + 6 \\
18 & = 3 \cdot 6 + 0
\end{align*}
\]

Thus \((24, 138) = 6\) by the Euclidean algorithm.
Thus (227, 143) = 1 by the Euclidean algorithm.

3. Problem 1.2.3

**Claim.** If \(a \mid b \) and \(b \mid c \), then \(a \mid c \)

**Proof.** Since \(a \mid b \), there exists \(q \in \mathbb{Z} \) such that \(b = q \cdot a \). Similarly, since \(b \mid c \), there exists \(r \in \mathbb{Z} \) such that \(c = r \cdot b \). Thus \(c = r \cdot (q \cdot a) = (r \cdot q) \cdot a \) (by associativity) so \(a \mid c \).

4. Problem 1.2.4 (a)

**Claim.** If \(a \mid b \) and \(a \mid c \), then \(a \mid (b + c) \)

**Proof.** Since \(a \mid b \) we can write \(b = q \cdot a \), and since \(a \mid c \) we can also write \(c = r \cdot a \). Thus \(b + c = q \cdot a + r \cdot a = (q + r) \cdot a \) and therefore \(a \mid (c + d) \).

5. Problem 1.2.7

Prove or disprove: If \(a \mid (b + c) \), then \(a \mid b \) or \(a \mid c \).

**Proof.** The above statement is false and we disprove it by a counterexample (any other method of proof is probably not the best idea, and more work than necessary if at all possible): For example, \(2 \mid (1 + 5) \) but \(2 \nmid 1 \) and \(2 \nmid 5 \).

6. Problem 1.2.14 Find the smallest positive integer in the given set:

(a) \( \{6u + 15v \mid u, v \in \mathbb{Z} \} \)

(b) \( \{12r + 17s \mid r, s \in \mathbb{Z} \} \)

(a) We can use the Euclidean algorithm to determine \((6, 15) = 3 \). Thus by Hungerford, theorem 1.3 we have that 3 is the smallest positive integer in this set.

(b) As in part (a) we find that \((12, 17) = 1 \), and therefore the smallest positive integer in this set is 1.

7. Problem 1.2.20

Prove or disprove each of the following statements:

(a) If \(2 \nmid a \), then \(4 \mid (a^2 + 1) \).

(b) If \(2 \nmid a \), then \(8 \mid (a^2 + 1) \).
Proof. Both of these statements are true. For (a) we see that if $2 \nmid a$ then $a$ is odd, that is $a = 2n + 1$. Then

\[
(a^2 + 1) = (a + 1)(a - 1) = (2n + 2)(2n) = 4(n + 1)n
\]

Part (b) is only slightly more difficult. From part (a) we know that $(a^2 + 1) = 4(n + 1)n$. The point here is to notice that one of any two consecutive integers must be even. Then either $n$ or $(n + 1)$ is even in which case we get another factor of 2 to divide $(a^2 + 1)$. Since $8 = 4 \cdot 2$ we’re done!