1. Prove that $n! > n^2$ for all integers $n \geq 4$.

**Solution** Proof by induction. Base case: when $n = 4$, $n! = 24$, while $n^2 = 16$, so this checks out. Now, suppose that $n > 4$ and that the statement is true for all $k$ where $4 \leq k < n$. Then,

$$n! = n((n - 1)!) > n((n - 1)^2) = n^3 - 2n^2 + n = n^2(n - 2) + n > n^2,$$

as desired.

2. Let $X$ be a finite set with $n$ elements. Determine, with proof, how many binary equivalence relations there are on $X$.

**Solution** A binary relation on $X$ is just a subset of $X \times X$. The subsets of $X \times X$ are the elements of the power set $P(X \times X)$. The set $X \times X$ has $n^2$ elements, so the set $P(X \times X)$ has $2^{(n^2)}$ elements. Therefore, there are $2^{(n^2)}$ binary relations on $X$.

3. How many rearrangements of MATHEMATICS are there where the Ms are not next to each other?

**Solution** In general, there are a total of

$$\frac{11!}{2!2!2!}$$

rearrangements of MATHEMATICS. Let $\Phi = \text{MM}$. Then, there are

$$\frac{10!}{2!2!}$$

rearrangements of $\Phi \text{ATHEATICS}$. These correspond to the rearrangements of MATHEMATICS in which the Ms are next to each other. So, there are

$$\frac{11!}{2!2!2!} - \frac{10!}{2!2!}$$

rearrangements of MATHEMATICS where the Ms are not next to each other.
4. Let’s play Canasta! The deck consists of 2 standard packs of 52 cards, 13 in each of 4 suits. So, there are 2 of every card, but we can’t tell the two copies apart. For example, there are 2 Aces of Hearts. How many different 5-card hands are there that contain only Hearts?

**Solution** First, suppose that the hand contains no duplicates; e.g., there are not 2 Aces of Hearts in the hand. Then, there are \( \binom{13}{5} \) such hands. Now, suppose that a single card is duplicated. There are 13 choices for the duplicated card, and \( \binom{12}{4} \) choices for the other cards. If 2 cards are duplicated, there are \( \binom{13}{2} \) choices for those cards and \( \binom{11}{1} \) choices for the other card. Therefore, there are

\[
\binom{13}{5} + \binom{13}{1} \binom{12}{4} + \binom{13}{2} \binom{11}{1}
\]

different flushes of Hearts.

5. Let \( X = \{1, 2, 3, 4, 5\} \). How many strings of length 1000 on \( X \) are there such that there are no substrings from \( \{1, 2\} \) of length more than 1.

**Solution** Let \( a_n \) be the number of string of length \( n \) on \( X \) such that there are no substring from \( \{1, 2\} \) of length more than 1. Then, \( a_0 = 1 \) and \( a_1 = 5 \). We find a recursive formula for the \( a_n \). Given any string \( t \) of length \( n - 1 \) on \( X \) of the same type, the strings \( 3t \), \( 4t \), and \( 5t \) are all of the appropriate type. Similarly, given any string \( t \) of length \( n - 1 \) on \( X \) of this type, the strings \( 13t \), \( 14t \), \( 15t \), \( 23t \), \( 24t \), and \( 25t \) are of the correct type. Thus, we see that

\[
a_n = 3a_{n-1} + 6a_{n-2}.
\]

To solve this, we consider the equation \( t^2 - 3t - 6 \). Using the quadratic formula, this has solutions \( r_1 = \frac{3 + \sqrt{33}}{2} \) and \( r_2 = \frac{3 - \sqrt{33}}{2} \). Solving the system of equations

\[
a + b = 1 \\
ar_1 + br_2 = 5,
\]

we find that \( a = \frac{7}{\sqrt{33}} \) and \( b = 1 - \frac{7}{\sqrt{33}} \). Therefore, there are

\[
\frac{7}{\sqrt{33}} \left( \frac{3 + \sqrt{33}}{2} \right)^{1000} + \left( 1 - \frac{7}{\sqrt{33}} \right) \left( \frac{3 - \sqrt{33}}{2} \right)^{1000}
\]

such strings.

6. Prove that in any set of 51 positive integers less than 100, there are two whose sum is 100.
Solution  Let \( a_1, \ldots, a_{51} \) be 51 positive integers less than 100. Let \( b_n = 100 - a_n \), for \( 1 \leq n \leq 51 \). First, note that \( b_n = a_n \) if and only if \( a_n = 50 \). If some \( a_n \) is equal to 50, then discarding \( a_n \) and \( b_n \), the rest of the numbers form 100 integers between 1 and 99. Thus, two of them are equal by the pigeonhole principle. So, \( a_k = b_j = 100 - a_j \) for some \( k \neq j \). So, we’re done. If no \( a_n \) is equal to 50 then the same argument works.

7. Show that if \( G \) is a simple graph, then either \( G \) or \( \overline{G} \) is connected.

Solution  Assume that \( G \) is a simple disconnected graph. Let \( v_1, \ldots, v_k, k \geq 2, \) be a vertex from each connected component of \( G \). This means that every vertex of \( G \) can be connected to exactly one of the \( v_i \), and no \( v_i \) can be connected to any other. Let \( x \) and \( y \) be two vertices in the vertex set of \( G \). We show that they are connected by a path in \( \overline{G} \). First, if \( x \) and \( y \) are in different components in \( G \), then there is actually an edge between them in \( \overline{G} \), so they are certainly connected by a path in this case. Now, assume that \( x \) and \( y \) are in the same component of \( G \), say the \( v_1 \) component. Then, there is an edge \( e_1 \) from \( x \) to \( v_2 \) in \( \overline{G} \) and an edge \( e_2 \) from \( y \) to \( v_2 \) in \( \overline{G} \). Thus, the path \((x, e_1, v_2, e_2, y)\) in \( \overline{G} \). Therefore, in this case too \( x \) and \( y \) are connected. Therefore, \( \overline{G} \) is connected.

8. Show that if \( G \) is a simple graph with at least two vertices, then there are two vertices in \( G \) with the same degree.

Solution  Suppose that \( G \) has \( n \) vertices. Since \( G \) is simple, the degree of each vertex is between 0 and \( n - 1 \). If the graph is connected, then the degree of each vertex is between 1 and \( n - 1 \). By the pigeonhole principle, two vertices have the same degree. If the graph is not connected, there is no vertex of degree \( n - 1 \). Thus, the degree of each vertex is between 0 and \( n - 2 \). Again, by the pigeonhole principle, two vertices have the same degree.

9. Prove that every tree with at least two vertices is a bipartite graph.

Solution  Choose a root for the tree \( T \). Then, let \( X \) consist of the vertices of even level, and let \( Y \) be the vertices of odd level. Then, \( T \) is bipartite on \( X \) and \( Y \).

10. Prove that the number of nonisomorphic binary trees with \( n \) vertices is the \( n \)th Catalan number.

Solution  Denote by \( C_n \) this number. Then, \( C_0 \) is 1. We can construct all isomorphism classes of binary trees with \( n \) vertices by choosing the number of vertices \( k \) of the left branch of the root together with a binary tree on \( k \) vertices together with a binary tree on \( n - k - 1 \) vertices. Therefore,

\[
C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}.
\]
But, this is the same recurrence relation satisfied by the Catalan numbers with the same initial condition.