Math 61 : Discrete Structures
First Midterm
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You have 50 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.
1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

The number of functions \( f : \{1, 2, \ldots, 100\} \rightarrow \{1, 2, \ldots, 5\} \) is:

(A) \(100^5\);  \(\boxed{(B) \ 5^{100}}\);  \(\boxed{(C) \ \frac{100!}{5!}}\);  \(\boxed{(D) \ \frac{100!}{95!}}\);  \(\boxed{(E) \ \frac{100!}{95! \cdot 5!}}\).

The number of relations from \(\{1, 2, \ldots, 100\}\) to \(\{1, 2, \ldots, 5\}\) is:

(A) \(100^5\);  \(\boxed{(B) \ 5^{100}}\);  \(\boxed{(C) \ 2^{105}}\);  \(\boxed{(D) \ 2^{500}}\);  \(\boxed{(E) \ \frac{100!}{95!}}\).

The number of increasing functions \( f : \{1, 2, \ldots, 100\} \rightarrow \{1, 2, \ldots, 5\} \) is:

(A) \(100^5\);  \(\boxed{(B) \ 5^{100}}\);  \(\boxed{(C) \ \frac{100!}{95!}}\);  \(\boxed{(D) \ \frac{100!}{95! \cdot 5!}}\);  \(\boxed{(E) \ 0}\).

The number of injective (one-to-one) functions \( f : \{1, 2, \ldots, 100\} \rightarrow \{1, 2, \ldots, 5\} \) is:

(A) \(100^5\);  \(\boxed{(B) \ 5^{100}}\);  \(\boxed{(C) \ \frac{100!}{95!}}\);  \(\boxed{(D) \ \frac{100!}{95! \cdot 5!}}\);  \(\boxed{(E) \ 0}\).
2. Write down the answer to each question. You do NOT need to justify your answers. Also, you do not need to simplify expressions such as $2^6$, $6!$, $C(6, 3)$, etc.

(a) Consider the function $f : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ given by $f(x) = 5 - x$.

(2 points) Is $f$ bijective? If so, what is the inverse of $f$?

Yes. $f^{-1} = f$.

(2 points) Viewing $f$ as a relation, write down the matrix associated to $f$.

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

(b) (2 points) Consider the following sequence of numbers:

$(5, 4, 8, 2)$.

Write down all its possible increasing subsequences.

$(5, 8), (4, 8), (5), (4), (8), (2), ()$. 
(c) Out of 50 people we need to select a committee composed of one president and five other (unordered) vice-presidents.

(2 points) How many selections are possible?

\[ 50 \cdot C(49, 5) \]

(2 points) Among the 50 people are you and your brother. How many selections are possible in which you become president AND your brother becomes a vice-president?

\[ C(48, 4) \]

(2 points) How many selections are possible in which either you become president, OR your brother becomes a vice-president, OR both?

\[ C(49, 5) + 49 \cdot C(48, 4) - C(48, 4). \]
3. Consider the following relation on \( \mathbb{R} \), the set of all real numbers:

\[ aRb \iff a \cdot b = 0. \]

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

(a) Is \( R \) a function?
No, because for example \( 0R0 \) and \( 0R1 \).

(b) Is \( R \) reflexive?
No, because for example \( 1 \cdot 1 \neq 0 \).

(c) Is \( R \) symmetric?
Yes, because \( a \cdot b = 0 \) implies \( b \cdot a = 0 \).

(d) Is \( R \) antisymmetric?
No: for example \( 1 \cdot 0 = 0 \cdot 1 = 0 \) but \( 0 \neq 1 \).

(e) Is \( R \) transitive?
No: for example \( 1 \cdot 0 = 0 \cdot 1 = 0 \) but \( 1 \cdot 1 \neq 0 \).
4. Prove by induction on \( n \) that:

\[
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n},
\]

for any integer \( n \geq 1 \).

In the base case \( n = 1 \), we have

\[
\frac{1}{2} = 1 - \frac{1}{2}.
\]

For the inductive step, suppose the equation is true for \( n \). For \( n + 1 \) we have

\[
\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \quad \text{(by the inductive hypothesis)}
\]

\[
= \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right) - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2}
\]

\[
= 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n+1} - \frac{1}{2n+2},
\]

so the statement is true by induction.