**Binomial coefficients**

The binomial coefficient \( \binom{n}{k} \) is used to count the number of \( k \)-element subsets of an \( n \)-element set. The term binomial coefficient comes from the binomial theorem:

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.
\]

Various properties involving binomial coefficients can be proved by using the above relationship for appropriate choices of \( a \) and \( b \). Binomial coefficients satisfy some important properties:

\[
\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{k} = \binom{n}{n-k} = \binom{n-1}{k-1} + \binom{n-1}{k},
\]

\[
\sum_{k=0}^{n} \binom{i}{k} = \binom{n+1}{i+1}.
\]

The binomial coefficients can be arranged in a triangular pattern known as Pascal's triangle. The first few rows of Pascal's triangle are given below.

<table>
<thead>
<tr>
<th>( \binom{n}{k} )</th>
<th>( k=0 )</th>
<th>( k=1 )</th>
<th>( k=2 )</th>
<th>( k=3 )</th>
<th>( k=4 )</th>
<th>( k=5 )</th>
<th>( k=6 )</th>
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<td>15</td>
<td>20</td>
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</tbody>
</table>

A combinatorial proof is used to show that \( "A = B" \) by counting some collection of objects in two different ways. For example, there are \( 2^n \) subsets of an \( n \)-element set but there are also \( \binom{2^n}{k} \) subsets with exactly \( k \) elements since the possible size of a subset is \( 0,1,\ldots,n \) we have by the addition rule that there are \( \sum_{k=0}^{2^n} \binom{k}{2^n} \) subsets; combining we have \( 2^n = \sum_{k=0}^{n} \binom{n}{k} \).

**Examples:**

1. Show for \( n \geq 1 \) that

\[
0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.
\]

Use this to show that the number of subsets with an even number of elements is equal to the number of subsets with an odd number of elements for \( n \geq 1 \).

2. Show for \( n \geq 0 \) that

\[
\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k+1} = \frac{1}{n+1}.
\]

(Hint: use integration between \(-1\) and \(0\) on some form of the binomial equation.)

3. Give a proof for the following identity for \( n \geq 0 \)

\[
3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}.
\]

(a) by using the binomial theorem.

(b) by counting the number of 2-colored subsets of an \( n \)-element set. (A 2-colored subset is a subset where each element has been assigned one of two colors, so for instance there are 9 2-colored subsets of \( \{1,2\} \), they are \( \emptyset \), \( \{1\} \), \( \{2\} \), \( \{1,2\} \), \( \{1,2\} \), \( \{1,2\} \) and \( \{1,2\} \).

4. The triangular number \( T_n \) is the number of tennis balls needed to form a triangle with \( n \) balls in the first row, \( n-1 \) balls in the second row, \( \ldots \), and 1 ball in the top row. Find an expression for \( T_n \) in terms of binomial coefficients.

5. The tetrahedron number \( \Delta_n \) is the number of tennis balls needed to form a tetrahedron with \( T_n \) balls in the first layer, \( T_{n-1} \) balls in the second layer, \( \ldots \), and \( T_1 \) balls in the top layer. Find an expression for \( \Delta_n \) in terms of binomial coefficients.

**Pigeon hole principle**

The pigeon hole principle says that if you are putting \( r \) objects into \( n \) different groups then one of the groups has at least \( \lceil r/n \rceil \) objects. This is used to show that two objects must be similar. Note that while this can be used to show that there are two or more objects which are in the same group we cannot say which group has multiple objects.

Proof by contradiction is a powerful technique where we show that something is true by first assuming the opposite holds and then end up with a contradiction (i.e., showing that the opposite cannot hold).

**Examples:**

1. If we pick 13 numbers between 1 and 20 show that there are two which differ by exactly 5.

2. If we pick 13 numbers between 1 and 20 show that there are two which differ by exactly 6.

3. Show it is possible to pick 13 numbers between 1 and 20 so that no two differ by exactly 7.

4. You and your three friends see a gumball machine which has five different colors of gumballs. You decide to keep buying gumballs until everyone has six gumballs of a single color (it is allowed that two different people have the same color of gumballs). What is the smallest number \( n \) of gumballs you will need to buy in order to guarantee that everyone gets their gumballs?

(Your answer will consist of two parts, first show that you can always satisfy the condition with \( n \).)
gumballs, while with \( n - 1 \) it is possible that you might not satisfy the condition.)

Recursion

A recurrence relationship on a sequence \( a_n \) is a relation-
ship between \( a_n \) and the previous terms, i.e., \( a_i \) for \( i < n \). An example is the minimal number of moves
needed to move \( n \) discs in the Tower of Hanoi problem
\( t_n \), by moving the top \( n - 1 \) discs, then the bottom
disc and the top \( n - 1 \) discs again, so \( t_n = 2t_{n-1} + 1 \).

An important sequence is the Fibonacci numbers
which are defined recursively by \( F_1 = 1, F_2 = 1 \) and
\( F_n = F_{n-1} + F_{n-2} \) (i.e., to get the next Fibonacci
number add the two most recent ones). This gives
the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ….

Examples:

1. Let \( q_n \) be the number of compositions (i.e., write
\( n \) as a sum of ordered nonnegative integers, or
“part”) of \( n \) with each part \( \geq 2 \). For example
\( q_5 = 5 \) since \( 6 = 4 + 2 = 2 + 4 = 3 + 3 = 2 + 2 + 2 \).

Give a recurrence for \( q_n \). Show \( q_n = F_{n-1} \) for
\( n \geq 2 \).

2. Let \( w_n \) be the number of binary sequences with no
000. Find a recurrence for \( w_n \). Give the numbers
\( w_1, w_2, …, w_7 \). These numbers are known as the
tribonacci numbers.

3. Consider the recurrence relationship
\[
d_{n+2} = \frac{1 + d_{n+1}}{a_n}, \quad \text{for } n \geq 1.
\]

Given \( d_1 = x \) and \( d_2 = y \) with \( x, y > 0 \) find \( d_{5583975} \)
in terms of \( x \) and \( y \).

Solving recursions

To solve a recurrence is to find an explicit expression
for \( a_n \) which depends only on \( n \) (i.e., you do not need
to know any of the previous terms). For example in
the Tower of Hanoi problem we have the recurrence
\( t_n = 2t_{n-1} + 1 \) and initial condition \( t_1 = 1 \), this has solution \( t_n = 2^n - 1 \).

We can solve recurrences by either looking for a pattern
and then verifying our guess using induction, i.e.,
showing that our guess satisfies the initial conditions
and the recursion relation; or by a systematic method.
The latter case applies for linear homogeneous recur-
rence relations with constant coefficients of order \( k \),
i.e., a recursion which can be written as
\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},
\]
where \( c_1, …, c_k \) are constants (i.e., independent of \( n \)).
The method to solve these is to first translate this into a
polynomial
\[
r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k.
\]
Then solve for the roots \( r_1, r_2, …, r_k \) of this poly-
nomial, either by factoring or the quadratic equation. If
these roots are distinct then
\[
a_n = D_1 r_1^n + D_2 r_2^n + \cdots + D_k r_k^n,
\]
where \( D_1, D_2, …, D_k \) are constants which are deter-
mined by the initial conditions. In the case of re-
peted roots the process is similar except now instead
of using \( D_1 r_1^n + D_2 r_1^n + \cdots + D_k r_1^n \) we intro-
duce powers of \( n \) to distinguish solutions so we have
\[
D_1 r_1^n + D_2 n r_1^n + \cdots + D_k n^{k-1} r_k^n.
\]

Occasionally we can use substitution to transform a
recursion into something which we can use the above
techniques on. Finally, if nothing else works we can
always look at the first few terms and try to find a
pattern.

Examples:

1. Solve the recurrence relationship
\[
r_{n+3} = 6r_{n+2} - 11r_{n+1} + 6r_n, \quad \text{for } n \geq 0
\]
with initial conditions \( r_0 = 5, r_1 = 6 \) and \( r_2 = 10 \).

2. Solve the recurrence relationship
\[
P_n = 2P_{n-1} + P_{n-2}, \quad \text{for } n \geq 2
\]
with initial conditions \( a_0 = 0 \) and \( a_1 = 1 \). These
numbers are known as the Pell numbers.

3. Solve the recurrence relationship
\[
\frac{S_n}{n} = S_{n-1} + \frac{2}{n}, \quad \text{for } n \geq 3
\]
with initial condition \( S_3 = 1 \).

4. Solve the recurrence relationship
\[
\frac{S_n}{n} = \frac{1}{n} \text{ if } n \text{ is even}, \quad \text{and 1 if } n \text{ is odd}
\]
with initial conditions \( S_0 = 0 \) and \( S_1 = 0 \). (Here
it is time to rewrite the recurrence into a single
equation.)
5. The initial conditions are not always consecutive. Sometimes we might now information about the behavior at the ends (or boundary) and we are trying to extrapolate information about what is happening between. Solve the following recursion

\[ c_n = 4c_{n-1} - 4c_{n-2}, \quad n \geq 1 \]

with boundary conditions \( c_0 = 5 \), and \( c_5 = 17 \).

---

**Graphs**

A graph \( G = (V, E) \) consists of two sets, a vertex set \( V \) and an edge set \( E \). The edge set either satisfies \( E \subseteq V^2 \) (i.e., two-element subsets where order does not matter so edges are not directed) or \( E \subseteq V \times V \) (i.e., two-element lists where order does matter so edges are directed). Pictorially vertices are represented by points "@" and edges by lines "\( \overrightarrow{\text{a-b}} \)" (when it is undirected) or arrows "\( \overrightarrow{\text{a-b}} \)" (when it is directed).

Graphs can be used to model many interactions (we already used directed graphs to model relations). A simple graph is an undirected graph without loops (an edge that goes from a vertex and returns to itself) or parallel edges (two or more edges going between the same vertices).

A graph \( G = (V, E) \) is bipartite if we can split \( V \) into two disjoint sets \( V = V_1 \cup V_2 \) so that all edges connect a vertex in \( V_1 \) to a vertex in \( V_2 \).

Some important simple graphs are the complete graph on \( n \) vertices, \( K_n \), which has \( n \) vertices and all \( \binom{n}{2} \) possible edges; the complete bipartite graph \( K_{m,n} \) which has \( m + n \) vertices in two parts, one part with \( m \) vertices and the other with \( n \) vertices and all \( mn \) possible edges between these two parts; the hypercube \( H_n \) or \( Q_n \) with vertices all possible \( 2^n \) binary strings and the \( n2^{n-1} \) edges connect two strings which differ in exactly two places.

\[ K_6 = \begin{array}{c} \text{Graph images} \end{array} \quad K_{3,4} = \begin{array}{c} \text{Graph images} \end{array} \quad Q_3 = \begin{array}{c} \text{Graph images} \end{array} \]

**Examples:**

1. Show that the hypercube \( Q_n \) is bipartite.
2. Consider a graph \( T_n \) where vertices are "binary" words of length \( n \) (i.e., a string of length \( n \) using the letters \( \{0, 1, 2\} \) and edges connect two words which differ in one entry. How many vertices does \( T_n \) have? How many edges does \( T_n \) have?
3. Continuing from the previous exercise, draw \( T_0 \), \( T_1 \) and \( T_2 \).
4. Given a graph \( G = (V, E) \), the line graph \( L(G) = (E, F) \) is the graph where each edge of \( G \) is a vertex of \( L(G) \) and two vertices in \( L(G) \) are adjacent if the corresponding edges are incident to the same vertex in \( G \). How many vertices are in \( L(K_n) \)? How many edges are in \( L(K_n) \)? Draw \( L(K_4) \).

---

**Eulerian graphs**

Two vertices are adjacent if there is an edge connecting them, an edge and a vertex are incident if the edge connects to the vertex. In an undirected graph the degree of a vertex \( v \), denoted \( \delta(v) \) or \( d(v) \), is the number of edges incident to \( v \) (loops count twice). An important fact about the degrees is

\[ \sum_{v \in V} \delta(v) = 2(\# \text{ of edges}). \]

One consequence of this is that there must be an even number of vertices with odd degree.

A graph is regular if all vertices have the same degree. One famous example of a regular graph is the Petersen graph on ten vertices which is regular of degree three (shown below).

\[ \begin{array}{c} \text{Graph image} \end{array} \]

A walk (or in some books a path) of length \( n \) is a sequence of vertices \( (v_0, v_1, \ldots, v_n) \) such that \( v_{i-1} \) is adjacent to \( v_i \) for \( i = 1, \ldots, n \). We say the walk is closed if it also satisfies \( v_0 = v_n \). A graph is connected if between any two vertices there is a path connecting them.

A subgraph of \( G = (V, E) \) is a graph \( G' = (V', E') \) with \( V' \subseteq V \) and \( E' \subseteq E \). An induced subgraph of \( G = (V, E) \) is a graph \( G' = (V', E') \) with \( V' \subseteq V \) and \( E' \subseteq E \) where all possible edges are taken. The components of \( G \) are the maximal connected induced subgraphs (i.e., the connected pieces of \( G \)).

An Eulerian cycle on a graph is a closed walk that uses each edge of the graph exactly once. A graph has an Eulerian cycle if and only if it is connected and the degree of each vertex is even.

**Examples:**

1. A bridge in a connected graph is an edge whose removal makes the graph not connected. Show that if a graph is connected and regular of degree \( r \), with \( r \) even, then the graph has no bridge.
2. Give an example of a connected graph which is regular of degree three which has a bridge.
3. Which of the following list of degrees is possible for a simple graph on six vertices. (If it is possible draw a graph that corresponds to it, while if it is not possible explain why it is not possible.)
   (a) \( 5, 4, 3, 2, 2, 1 \)
   (b) \( 3, 3, 3, 1, 1, 1 \)
4. Show that if a graph is bipartite all closed walks have even length.

5. Show that if all the closed walks in a graph have even length then the graph is bipartite. (You may assume that the graph is connected.)

6. A domino is a rectangle divided into two squares, each square having between 0 and 6 pips (it is possible for both squares to have the same number of pips). A full set of dominoes is one where each possible domino occurs once. Show that the dominoes can be arranged in a circular pattern so that for two adjacent dominoes the touching squares have the same number of pips.

7. Show that if the dominoes instead have between 0 and 9 pips on each square that it is impossible to arrange a full set of dominoes in a circular pattern so that for two adjacent dominoes the touching squares have the same number of pips.

8. What is the smallest number of dominoes that have to be removed from the complete set in the previous problem so that the remaining dominoes can now be arranged in a circular fashion?

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** ALSO:**

**Hamiltonian graphs**

A Hamiltonian cycle is a closed walk which visits each vertex exactly once. (Contrast this with Eulerian cycles which visit each edge exactly once.) To prove that a graph is Hamiltonian we only need to show the closed walk visiting each vertex exactly once; on the other hand to show that a graph is not Hamiltonian takes an argument. The graph $K_n$ is Hamiltonian for all $n$; $K_{n,n}$ is Hamiltonian if and only if $n = n$; the Petersen graph is not Hamiltonian.

The hypercube $Q_n$ is Hamiltonian for all $n$. A Hamiltonian walk in a hypercube is a Gray code (an arrangement of the binary words of length $n$ so that any two consecutive words differ in exactly one entry.

**Adjacency matrix**

We can use the adjacency matrix $A$ to represent a graph. The rows and columns are indexed by the vertices and the entries indicate adjacency relationships. Namely we have

$$ A = \begin{pmatrix} \ldots & a_{i,j} & \ldots \\ \vdots & \ddots & \vdots \\ \ldots & \ldots & \end{pmatrix} $$

where $a_{i,j}$ counts the number of edges joining the $i$th vertex to the $j$th vertex (for simple graphs this will be a 0-1 matrix). The graph $A$ is not unique since reordering vertices can give a different matrix.

If the graph is undirected the matrix $A$ is symmetric; the row sums and the column sums give the degrees. If the graph is directed the matrix $A$ does not have to be symmetric; the row sums give the out-degrees and the column sums give the in-degrees. If the matrix is of the form

$$ A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}, $$

where $B^T$ is the transpose of $B$, then the graph is bipartite.

Adjacency matrices are useful to count the number of walks joining two vertices. Namely we have

$$ (A^n)_{i,j} = \# \text{ walks joining } i \text{ and } j. $$

The diagonal of $A^3$ counts the degree. The trace of $A^3$ is equal to six times the number of triangles.

A related matrix is the incidence matrix $Q$ which is a 0-1 matrix with the rows indexed by the vertices and columns indexed by the edges. The entry $q_{i,j}$ is 1 if and only if the corresponding vertex is incident to the corresponding edge. If $G$ is a simple graph then $Q^T Q = \Delta - A$ where $\Delta$ is the diagonal degree matrix with $\Delta_{ii} = d_i$, where $d_i$ is the degree of $v_i$.
More counting

Selecting \( r \) elements out of \( n \) objects when order matters is called an \( r \)-permutation of \( n \). This can be done in 
\[
P(n,r) = \frac{n(n-1) \cdots (n-r+1)}{r!} = \frac{n!}{(n-r)!}
\]
different ways. Selecting \( r \) elements out of \( n \) objects when order doesn’t matter is called an \( r \)-combination of \( n \). This can be done in 
\[
C(n,r) = \binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{r(r-1) \cdots 1} = \frac{n!}{r!(n-r)!}
\]
different ways. Table of small valued \( \binom{n}{r} \).

<table>
<thead>
<tr>
<th>( \binom{n}{k} )</th>
<th>( k=0 )</th>
<th>( k=1 )</th>
<th>( k=2 )</th>
<th>( k=3 )</th>
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</table>

The number of ways to arrange \( n_1 \) objects of type 1, \( n_2 \) objects of type 2, \ldots, and \( n_k \) objects of type \( k \) is 
\[
\frac{(n_1+n_2+\cdots+n_k)!}{n_1!n_2!\cdots n_k!}
\]

*Bars and Stars.* The number of ways to divide \( n \) identical objects into \( k \) distinct sets (some of which might be empty) is 
\[
\binom{n+k-1}{k-1} = \binom{n+k-1}{n}
\]

Examples:

1. A magazine editor is laying out a photo spread. She has to put six pictures in out of the 73 that were shot in the studio. How many different ways can she arrange the layout?

2. At Brunette’s pizza there are seventeen different toppings. How many different four topping pizzas are available (toppings cannot be repeated)?

3. A normal deck of cards consists of 52 different cards. How many different seven card hands are there?

4. How many ways are there to rearrange the letters of the word “MISSISSIPPI”?

5. How many ways are there to rearrange the letters of the word “MISSISSIPPI” if there cannot be two consecutive S’s?

6. There are two math majors and three computer science majors dividing the leftover free food from a talk. They count and discover there are 9 cookies and 34 jelly beans. How many ways are there to divide the cookies and jelly beans among the five students?

7. Continuing the previous problem, after some tense negotiation they agree that every math major will get at least two cookies and every computer science major will get at least six jelly beans. Now how many ways are there to divide the cookies and jelly beans?

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**ALSO:**

- **TRAVELING SALESMAN**
- **SHORTEST PATH PROBLEM**
  - (DIJKSTRA’S ALGORITHM)