31/B - Final - Solutions

December 9, 2011

1. (20 points) Calculate \( g'(1) \) and \( g''(1) \), where \( g(x) \) is the inverse of \( f(x) = x + \ln x \).

Solution First we solve, \( 1 = x + \ln x \), and we see that \( x = 1 \). Thus, \( g(1) = 1 \). Now, \( f'(x) = 1 + \frac{1}{x} \). So,

\[
g'(1) = \frac{1}{f'(g(1))} = \frac{1}{1 + \frac{1}{g(1)}} = \frac{1}{1 + \frac{1}{1}} = \frac{1}{2}.
\]

2. (20 points) Evaluate the integral

\[ \int x\sqrt{9 - x^2} \, dx \]

using trigonometric substitution.

Solution We substitute \( x = 3 \sin \theta \). Then, \( dx = 3 \cos \theta \, d\theta \). So, the integral becomes

\[
\int x\sqrt{9 - x^2} \, dx = \int 3 \sin \theta \sqrt{9 - 9 \sin^2 \theta} \cos \theta \, d\theta
= 27 \int \sin \theta \cos^2 \theta \, d\theta.
\]

Now we substitute \( u = \cos \theta \). Then, \( du = -\sin \theta \, d\theta \), so the integral becomes

\[
27 \int \sin \theta \cos^2 \theta \, d\theta = -27 \int u^2 \, du = -27 \frac{u^3}{3} = -9 \cos^3 \theta.
\]

Using triangles, we see that

\[
\cos \theta = \frac{\sqrt{9 - x^2}}{3}.
\]
Thus, the final answer is
\[
\int x\sqrt{9-x^2} \, dx = -\frac{9 (9-x^2)^{3/2}}{27} = -\frac{(9-x^2)^{3/2}}{3}.
\]

3. **(20 points)** Evaluate the integral

\[
\int \frac{x^5 + 2}{x^2(x + 1)} \, dx.
\]

**Solution** First, dividing \(x^5 + 2\) by \(x^3 + x^2\) we see that \(x^5 + 2 = (x^2 - x + 1)(x^3 + x^2) - x^2 + 2\). So,

\[
\int \frac{x^5 + 2}{x^2(x + 1)} \, dx = \int \frac{(x^2 - x + 1)(x^3 + x^2) - x^2 + 2}{x^3 + x^2} \, dx
\]

\[
= \int (x^2 - x + 1) \, dx - \int \frac{x^2 - 2}{x^3 + x^2} \, dx
\]

\[
= \frac{x^3}{3} - \frac{x^2}{2} + x - \int \frac{x^2 - 2}{x^2(x + 1)} \, dx.
\]

We solve for \(A\), \(B\), and \(C\) in the equation

\[
\frac{x^2 - 2}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}
\]

by multiplying across by \(x^2(x + 1)\) to obtain

\[
x^2 - 2 = Ax(x + 1) + B(x + 1) + Cx^2
\]

\[
= Ax^2 + Ax + Bx + B + Cx^2
\]

\[
= (A + C)x^2 + (A + B)x + B.
\]

Equating coefficients, we see that \(B = -2\), \(A = 2\), and \(C = -1\). Therefore,

\[
\int \frac{x^2 - 2}{x^2(x + 1)} \, dx = \int \frac{2 \, dx}{x} - \int \frac{2 \, dx}{x^2} - \int \frac{dx}{x + 1}
\]

\[
= 2 \ln|x| + \frac{2}{x} - \ln|x + 1| + C.
\]

Thus, the final answer is

\[
\int \frac{x^5 + 2}{x^2(x + 1)} \, dx = \frac{x^3}{3} - \frac{x^2}{2} + x - 2 \ln|x| - \frac{2}{x} + \ln|x + 1| + C.
\]

4. **(20 points)** Evaluate the integral

\[
\int \sin(ln \, x) \, dx.
\]
Solution There are two ways to do this problem. One, you may simply start with integration by parts with \( u = \sin(\ln x) \). Two, you may first substitute. I show the second way here. First, we must substitute \( w = \ln x \). Then, \( dw = \frac{dx}{x} \), or \( dx = xdw = e^w dw \). Thus, the integral becomes
\[
\int \sin(\ln x) \, dx = \int e^w \sin(w) \, dw.
\]
Second, we do integration by parts twice both times with \( u = e^w \) to obtain
\[
\int e^w \sin(w) \, dw = -e^w \cos(w) + \int e^w \cos(w) \, dw = -e^w \cos(w) + e^w \sin(w) - \int e^w \sin(w) \, dw
\]
Therefore,
\[
\int \sin(\ln x) \, dx = \int e^w \sin(w) \, dw = \frac{e^w}{2} (\sin(w) - \cos(w)) = \frac{x}{2} \left( \sin(\ln x) - \cos(\ln x) \right).
\]

5. (20 points) Determine whether or not the improper integral
\[
\int_1^2 \frac{dx}{x \ln x}
\]
converges.

Solution We do the substitution \( u = \ln x, \, du = \frac{dx}{x} \) to obtain
\[
\int_1^2 \frac{dx}{x \ln x} = \int_{\ln 1}^{\ln 2} \frac{du}{u} = \lim_{R \to 0} u \bigg|_{\ln R}^{\ln 2},
\]
which diverges.

6. (20 points) Use the error bound for Taylor polynomials to find a value of \( n \) for which
\[
| \ln 2 - T_n(2) | \leq 10^{-6},
\]
where \( T_n \) is the \( n \)th Taylor polynomial for \( f(x) = \ln x \) with center 1.

Solution We know that the \( n \)th derivative of \( \ln x \) is
\[
f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}.
\]
On The interval \([1, 2]\), the function \(|f^{(n)}(x)|\) is decreasing, so we can take \( K_n = |f^{(n)}(1)| = (n-1)! \). The error bound then gives,
\[
| \ln 2 - T_n(2) | \leq \frac{K_{n+1} (2-1)^{n+1}}{(n+1)!} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.
\]
So, we need
\[ \frac{1}{n+1} \leq \frac{1}{1000000}, \]
or \( n \geq 999999. \)

7. (20 points) Determine whether or not
\[ \sum_{n=1}^{\infty} \frac{5^{n^2}}{n!} \]
converges.

Solution First, note that \( 5^{n^2} = (5^n)^n \). Second, note that \( n! \leq n^n \) for all \( n \geq 1 \). Therefore,
\[ \sum_{n=1}^{\infty} \frac{5^{n^2}}{n!} \geq \sum_{n=1}^{\infty} \frac{5^{n^2}}{n^n}. \]
If we show the right-hand series diverges, then we will have shown that the left-hand series diverges by the comparison test. The root test gives
\[ L = \lim_{n \to \infty} \left( \frac{(5^n)^n}{n^n} \right)^{1/n} = \lim_{n \to \infty} \frac{5^n}{n} = +\infty. \]
So,
\[ \sum_{n=1}^{\infty} \frac{5^{n^2}}{n!} \]
diverges.

8. (20 points) Find the interval of convergence of the power series
\[ F(x) = \sum_{n=1}^{\infty} \frac{n(2x)^{2n}}{5n + 4}. \]

Solution The ratio test produces
\[ \rho(x) = \lim_{n \to \infty} \left| \frac{(n+1)(2x)^{2n+2}}{5(n+1)+4} \right| \]
\[ = \lim_{n \to \infty} \left| \frac{(2x)^{2n+2}}{(2x)^{2n}} \right| \frac{5n + 4 n + 1}{5n + 9} \]
\[ = (2|x|)^2 = 4|x|^2. \]
Therefore, $\rho(x) < 1$ when $|x|^2 < \frac{1}{4}$. That is, when $|x| < \frac{1}{2}$. Thus, the radius of convergence is $R = \frac{1}{2}$. When $x = -\frac{1}{2}$ or $x = \frac{1}{2}$ the limit of the sequence is not zero, so the divergence test says that the series diverges. Thus, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

9. **(20 points)** Approximate using Taylor series the integral

$$S = \int_0^1 \cos(x^3) \, dx$$

with an error of at most $10^{-4}$.

**Solution** The Taylor series for $\cos(x^3)$ is

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}.$$ 

Thus,

$$\int \cos(x^3) \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(6n + 1)(2n)!}.$$

So,

$$S = \int_0^1 \cos(x^3) \, dx = \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(6n + 1)(2n)!} \right) |_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(6n + 1)(2n)!}.$$

This is an alternating sum, and we know that if

$$S_N = \sum_{n=0}^{N} (-1)^n \frac{1}{(6n + 1)(2n)!},$$

then

$$|S - S_N| < \frac{1}{(6(N + 1) + 1)(2(N + 1))!}.$$ 

So, we need to find $N$ such that

$$(6N + 7)(2N + 2)! > 10000.$$ 

If $N = 1$, we have $(13)(4)! = 13 \cdot 24 = 312$. If $N = 2$, we have $(19)(6)! = 13680$. So, $N = 2$ works. That is,

$$S_2 = 1 - \frac{1}{7 \cdot 2!} + \frac{1}{13 \cdot 4!}$$

approximates the integral with an error of at most $10^{-4}$.

10. **(20 points)** Find the terms through degree 7 of the Taylor series $T(x)$ centered at $c = 0$ of $f(x) = \sin(x) \cos(x)$. 


Solution We know that the $T(x)$ is the product of the Taylor series centered at 0 of sin($x$) and cos($x$). That is,

$$T(x) = \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right) \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$$

$$= \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

$$= \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \right) - \frac{x^2}{2!} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} \right) + \frac{x^4}{4!} \left( x - \frac{x^3}{3!} \right) - \frac{x^6}{6!} (x) + \cdots$$

$$= x - \left( \frac{1}{3!} + \frac{1}{2!} \right) x^3 + \left( \frac{1}{5!} + \frac{1}{2! \cdot 3!} + \frac{1}{4!} \right) x^5 - \left( \frac{1}{7!} + \frac{1}{2! \cdot 5!} + \frac{1}{3! \cdot 4!} + \frac{1}{6!} \right) x^7 + \cdots$$