Math 31b : Integration and Infinite Series
Midterm 1

1. Consider the function \( f(x) = xe^x \).
   (a) (5 points) Find the derivative \( f'(x) \).

   **Solution.** We use logarithmic differentiation. Since \( \ln f(x) = \ln(xe^x) = e^x \ln x \), we get
   \[
   \frac{f'(x)}{f(x)} = (\ln f(x))' = (e^x \ln x)' = \frac{e^x}{x} + e^x \ln x
   \]
   so
   \[
   f'(x) = \left(\frac{e^x}{x} + e^x \ln x\right)xe^x.
   \]

   (b) (5 points) Let \( g = f^{-1} \) denote the inverse function. Find \( g'(1) \).

   **Solution.** We have \( f(1) = 1 \), so \( f^{-1}(1) = 1 \) and
   \[
   g'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(1)} = \frac{1}{(e + 1 \cdot 0)1} = \frac{1}{e}.
   \]

2. We have a sample consisting of 100 grams of a radioactive isotope. After one year it decays to 90 grams.
   (a) (5 points) What is the half-life of the isotope?

   **Solution.** The quantity must satisfy the exponential decay law \( P(t) = P_0e^{-kt} \). We have
   \[
   P(1) = 100e^{-k} = 90
   \]
   so the growth constant \( k \) satisfies \( e^{-k} = 9/10 \), i.e. \( k = -\ln(9/10) \). The half-life is \( -\frac{\ln 2}{\ln(9/10)} \).

   (b) (5 points) After how many years will the sample decay to 30 grams?

   **Solution.**
   \[
   P(t) = 30 \implies 100e^{-kt} = 30 \implies -kt = \ln(3/10). \quad \text{Therefore } t = \frac{\ln(3/10)}{\ln(9/10)}.
   \]

3. A function \( f(x) \) satisfies the differential equation
   \[
   f'(x) + 4f(x) = 2.
   \]
   If \( f(0) = 1 \), what is \( f(1) \)?

   **Solution.** We can rewrite the equation as \( f' = -4(f - 1/2) \), with solution
   \[
   f(t) = \frac{1}{2} + Ce^{-4t}
   \]
   for some \( C \). From \( f(0) = 1 \) we get \( C = \frac{1}{2} \) and
   \[
   f(1) = \frac{1 + e^{-4}}{2}.
   \]
4. Calculate the following limits:
   (a) (5 points)
   \[
   \lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}
   \]

   **Solution.** It is of the form 0/0. Using l'Hôpital we get
   \[
   \lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi} = \lim_{x \to \pi} \frac{(\cos x)e^{\sin x}}{1} = -1 \cdot e^0 = -1.
   \]

   (b) (5 points)
   \[
   \lim_{x \to 0^+} \frac{e^{-1/x^4}}{x}
   \]

   **Solution.** Substitute \( t = 1/x \). In terms of \( t \), the limit is
   \[
   \lim_{t \to \infty} te^{-t^4} = \lim_{t \to \infty} \frac{t}{e^{t^4}} = (\text{by l'Hôpital}) \lim_{t \to \infty} \frac{1}{4t^3e^{t^4}} = 0.
   \]

5. Calculate the definite integral:
   \[
   \int_0^{\sqrt{\frac{\pi}{4}}} x^3 \cos(2x^2) \, dx.
   \]

   **Solution.**
   Substitute \( 2x^2 = u \) so \( 4xdx = du \). We get
   \[
   \int_0^{\sqrt{\frac{\pi}{4}}} x^3 \cos(2x^2) \, dx = \int_0^{\pi/2} (u/2) \cos(u) \, \frac{du}{4} = \frac{1}{8} \int_0^{\pi/2} u \cos(u) \, du.
   \]
   Using integration by parts, the last expression equals
   \[
   \frac{1}{8} \left( u \sin(u) \right|_0^{\pi/2} - \int_0^{\pi/2} \sin(u) \, du \right) = \frac{1}{8} \left( \frac{\pi}{2} - 0 + \cos u \right|_0^{\pi/2} = \frac{1}{8} \left( \frac{\pi}{2} - 1 \right).
   \]