Math 31b : Integration and Infinite Series

Final Exam

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You have 180 minutes.
No books, notes or calculators are allowed.
Do not use your own scratch paper.
1. **Multiple Choice.** (2 points each) Circle the correct answer. You do not need to justify your answer, and no partial credit will be given.

   (i) The infinite series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

      - (a) converges to $e^2$
      - (b) converges to $e^{1/2}$
      - (c) converges to 2
      - (d) diverges to infinity
      - (e) None of the above.

   (ii) The interval of convergence of the power series $\sum_{n=1}^{\infty} n(x - 1)^n$ is

      - (a) \{1\}
      - (b) \((-1, 1)\)
      - (c) \([-1, 1)\)
      - (d) \((0, 2)\)
      - (e) \([0, 2)\)
      - (f) None of the above.

   (iii) If $\{a_n\}$ is a divergent sequence, then:

      - (a) $\{a_n\}$ is bounded
      - (b) $\{a_n\}$ is unbounded
      - (c) $\{a_n\}$ is monotonic
      - (d) $\{\frac{1}{a_n}\}$ converges
      - (e) None of the above need to be true.
(iv) For \( p > 1 \), the alternating \( p \)-series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \)

\( (a) \) converges absolutely

(b) converges conditionally

(c) diverges to infinity

(d) diverges, but not to infinity

(v) If \( \{a_n\} \) is a sequence of positive numbers such that \( \lim_{n \to \infty} a_n = 0 \), then

(a) \( \sum_{n=1}^{\infty} (-1)^n a_n \) must converge absolutely

(b) \( \sum_{n=1}^{\infty} (-1)^n a_n \) must converge conditionally

(c) \( \sum_{n=1}^{\infty} (-1)^n a_n \) must converge, but we can’t say if it does so absolutely or conditionally

(d) \( \sum_{n=1}^{\infty} (-1)^n a_n \) must diverge

\( (e) \) \( \sum_{n=1}^{\infty} (-1)^n a_n \) may converge or diverge.
2. Multiple Choice. (2 points each) You do not need to justify your answer, and no partial credit will be given.

Write a letter (a-j) in each box, indicating the Maclaurin series that corresponds to the function \( f(x) \):

(i) \( f(x) = x \sin x \) \( \boxed{h} \)

(ii) \( f(x) = x \cos x \) \( \boxed{f} \)

(iii) \( f(x) = \tan^{-1}(x^2) \) \( \boxed{e} \)

(iv) \( f(x) = \frac{1}{1 + x^2} \) \( \boxed{a} \)

(v) \( f(x) = \frac{1}{(1 + x)^2} \) \( \boxed{b} \)

You can choose from:

(a) \( \sum_{n=0}^{\infty} (-1)^n x^{2n} \) ;

(b) \( \sum_{n=0}^{\infty} (-1)^n (n + 1)x^n \) ;

(c) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \) ;

(d) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n + 1} \) ;

(e) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n + 1} \) ;

(f) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \) ;

(g) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n + 2)!} \) ;

(h) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n + 1)!} \) ;

(i) \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n + 2)!} \) ;

(j) None of these.
3. Evaluate the following limits:

(a) (5 points)
\[
\lim_{x \to 3} \frac{\sqrt{1 + x} - 2}{x - 3}
\]

(b) (5 points)
\[
\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^{x^2}
\]

**Solution.** (a) Using l'Hopital:
\[
\lim_{x \to 3} \frac{\sqrt{1 + x} - 2}{x - 3} = \lim_{x \to 3} \frac{1}{2\sqrt{1 + x}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}
\]

(b) Let \(f(x) = (1 - 1/x)^{x^2}\). Then \(\ln f(x) = x^2 \ln(1 - 1/x)\).

\[
\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln(1 - 1/x)}{1/x^2} = (\text{by l'Hopital}) \lim_{x \to \infty} \frac{\frac{1}{x^2}}{-2/x^3} = \lim_{x \to \infty} \frac{x - 1}{2} = -\infty
\]

so

\[
\lim_{x \to \infty} f(x) = e^{\lim_{x \to \infty} \ln f(x)} = e^{-\infty} = 0.
\]
4. (10 points) Evaluate the integral

\[ \int (\ln x)^2 \, dx \]

**Solution.** Substitute \( u = \ln x \) so \( x = e^u, \, dx = e^u \, du \). Then:

\[ \int (\ln x)^2 \, dx = \int u^2 e^u \, du \]

Using integration by parts twice, we get

\[
\int u^2 e^u \, du = u^2 e^u - \int 2ue^u \, du = u^2 e^u - 2ue^u + \int 2e^u \, du =
\]

\[ u^2 e^u - 2ue^u + 2e^u + C = (\ln x)^2 x - 2x \ln x + 2x + C. \]
5. (a) (5 points) Find the third Taylor polynomial $T_3(x)$ centered at 1 for the function $f(x) = \sqrt{x}$.

(b) (5 points) Estimate the error $|\sqrt{1.2} - T_3(1.2)|$. (You do not need to simplify your answer.)

Solution. (a) We have

$$f'(x) = (1/2)x^{-1/2}, \quad f''(x) = (1/2)(-1/2)x^{-3/2}, \quad f'''(x) = (1/2)(-1/2)(-3/2)x^{-5/2}$$

The values at $x = 1$ are

$$f(1) = 1, \quad f'(1) = 1/2, \quad f''(1) = -1/4, \quad f'''(x) = 3/8.$$

Hence

$$T_3(x) = 1 + \frac{1/2}{1!} (x - 1) + \frac{-1/4}{2!} (x - 1)^2 + \frac{3/8}{3!} (x - 1)^3$$

$$= 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3.$$

(b) $|f^{(4)}(x)| = |(1/2)(-1/2)(-3/2)(-5/2)x^{-7/2}| = |(15/16)x^{-7/2}|$. The maximum value of this on the interval $[1, 1.2]$ is $K_4 = 15/16$, so by the error bound for Taylor polynomials:

$$|\sqrt{1.2} - T_3(1.2)| \leq \frac{K_4 x^{(4)}(2)^4}{4!} = \frac{15}{16 \cdot 24 \cdot 5^4}.$$
6. (10 points) Consider the half-circle $S$ given by $x^2 + (y - 4)^2 = 1$ and $y \geq 4$. Find the area of the surface of revolution obtained by rotating $S$ around the $x$-axis.

**Solution.** We use the formula

$$S = 2\pi \int_{a}^{b} f(x) \sqrt{1 + (f'(x))^2} \, dx$$

for $f(x) = 4 + \sqrt{1 - x^2}$ and $a = -1, b = 1$. Note that $f'(x) = -x/\sqrt{1 - x^2}$ so

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + \frac{x^2}{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}}.$$

Therefore,

$$S = 2\pi \int_{-1}^{1} (4 + \sqrt{1 - x^2}) \cdot \frac{1}{\sqrt{1 - x^2}} \, dx = 2\pi(2 + 4 \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}).$$

Using the substitution $x = \sin \theta$ we get

$$S = 4\pi + 8\pi \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = 4\pi + 8\pi^2.$$
7. (10 points) Does the improper integral

\[
\int_0^1 \frac{dx}{x^7 + x}
\]

converge or diverge? Justify your answer carefully.

**Solution.** For \( x \in (0, 1) \) we have \( x^7 \leq x \) so \( x^7 + x \leq 2x \). Therefore,

\[
\int_0^1 \frac{dx}{x^7 + x} \geq \int_0^1 \frac{dx}{2x}
\]

The latter integral diverges by the \( p \)-test with \( p = 1 \). Hence, the original integral diverges as well, using the comparison test.
8. (5 points each) Do the infinite series

(a) 
\[ \sum_{n=1}^{\infty} \frac{1}{n \ln n + 1} \]

(b) 
\[ \sum_{n=1}^{\infty} \left( \frac{2n + \sqrt{n}}{n + 2\sqrt{n}} \right)^n \]

converge or diverge? Justify your answers carefully.

Solution. (a) Use the limit comparison test with \( a_n = 1/(n \ln n + 1) \) and \( b_n = 1/(n \ln n) \). Since \( \lim_{n \to \infty} a_n/b_n = 1 \), it suffices to study the convergence of \( \sum 1/(n \ln n) \). By the integral test, this is the same question as the convergence of the following improper integral, which can be evaluated using the substitution \( e^u = x \):
\[
\int_1^\infty \frac{dx}{x \ln x} = \int_{\ln 1}^{\infty} \frac{e^u du}{e^u u} = \int_{\ln 1}^{\infty} \frac{du}{u}
\]
This diverges. Therefore, the series diverges.

(b) Use the Root Test:
\[ \lim_{n \to \infty} \left( \frac{2n + \sqrt{n}}{n + 2\sqrt{n}} \right) = 2 > 1 \]
so the series diverges.
9. (5 points each) Evaluate the infinite series

(a) \[ \sum_{n=2}^{\infty} \frac{(-2)^n + 1}{3^n} \]

(b) \[ \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} \]

Solution. (a) Use the formula for geometric series:

\[
\sum_{n=2}^{\infty} \frac{(-2)^n + 1}{3^n} = \left(\frac{-2}{3}\right)^2 \sum_{n=0}^{\infty} (-2/3)^n + \left(\frac{1}{3}\right)^2 \sum_{n=0}^{\infty} (1/3)^n = \frac{4}{9} \cdot \frac{1}{1 - \frac{-2}{3}} + \frac{1}{9} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{4}{15} + \frac{1}{6} = \frac{13}{30}
\]

(b) Use the partial fractions decomposition

\[
\frac{1}{(n-1)(n+1)} = \frac{1}{2} \left( \frac{1}{n-1} - \frac{1}{n+1} \right)
\]

and then telescoping:

\[
\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} = \lim_{N \to \infty} \sum_{n=2}^{N} \frac{1}{(n-1)(n+1)}
\]

\[
= \lim_{N \to \infty} \frac{1}{2} \left( \frac{1}{1} - \frac{1}{N} + \frac{1}{N-2} - \frac{1}{N} + \frac{1}{N-4} - \frac{1}{N} + \cdots + \frac{1}{N-2} - \frac{1}{N} + \frac{1}{N-1} - \frac{1}{N} \right)
\]

\[
= \frac{1}{2} \lim_{N \to \infty} \left( \frac{1}{1} + \frac{1}{N} - \frac{1}{1} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{1} + \frac{1}{2} \right)
\]

\[
= \frac{3}{4}
\]
10. (a) (6 points) Find a power series $F(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfying the differential equation $F'' = -F$, with initial conditions $F(0) = 1, F'(0) = 1$.

(b) (3 points) For what values of $x$ does the series $F(x)$ converge? Justify your answer.

(c) (1 point) Write the function $F(x)$ in closed form—that is, as a simple expression such as $x \tan^{-1}(x), \cos(x^2)$, etc.

Solution.

(a) We have

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

$$F'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

$$F''(x) = 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2} + \cdots$$

The initial conditions become $a_0 = F(0) = 1, a_1 = F'(0) = 1$. Equating coefficients in $F = -F''$ we get $a_{n-2} = -n(n-1)a_n$ so

$$a_0 = 1, \ a_2 = -\frac{1}{1 \cdot 2}, \ a_4 = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}, \ \ldots, \ a_{2n} = (-1)^n \frac{1}{(2n)!}$$

$$a_1 = 1, \ a_3 = -\frac{1}{2 \cdot 3}, \ a_5 = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}, \ \ldots, \ a_{2n+1} = (-1)^n \frac{1}{(2n+1)!}$$

We get

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(b) The general term in $F(x)$ can be written as $\pm \frac{x^n}{n!}$. By the ratio test (where the sign doesn’t matter since we take absolute values)

$$\lim_{n \to \infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0$$

for all $x$, so $F(x)$ converges for all $x$.

(c) $F(x) = \sin x + \cos x$. 