1. In this problem, $R = \mathbb{C}[G]$ is the group ring over the complex numbers of a finite group $G$. All modules will be left modules. As a hint for much of the problem, take another look at problem 3 from the 110AH take home final (can be found on the web page for that class, linked through my home page). (2 pts each)

a) Prove that $R$ is both Noetherian and Artinian (left and right).

b) Let $M$ be a finitely generated $R$-module. Prove that there is a unique (up to isomorphism) direct sum decomposition $M \cong \bigoplus_i Q_i$ into a sum of finitely many simple $R$-modules.

c) Suppose $Q$ is a simple $R$-module. Show that $Q$ is finitely generated.

d) Let $P$ and $Q$ be non-isomorphic simple $R$-modules. Show that $\text{Hom}_R(P, Q) = 0$ and that there is an isomorphism of rings $\mathbb{C} \cong \text{End}_R(Q)$.

e) Now let $M$ and $N$ be two finitely generated $R$-modules. Prove that $M \cong N$ if and only if $M$ and $N$ have the same (up to isomorphism) simple modules with the same multiplicities in their direct sum decomposition.

f) Given two finitely generated $R$-modules $M$ and $N$, give a formula for $\dim_{\mathbb{C}}(\text{Hom}_R(M, N))$ in terms of the common simple summands in their respective direct sum decompositions.

g) Now let $G$ be an abelian group. Show: if $Q$ is a simple $R$-module, then the dimension of $Q$ as a $\mathbb{C}$-vector space is 1.

2. In this problem, $R$ is a commutative local ring with maximal ideal $m$ and residue field $k = R/m$. (2 pts each)

a) Suppose $M$ is a finitely generated $R$-module such that $mM = M$. Show that $M = 0$.

b) Show that for any $R$-module $M$, $M/mM$ is a $k$-vector space. Then use a) to show: if $M$ is a finitely generated $R$-module such that the $k$-vector space $M/m = 0$, then $M = 0$.

c) Now show that any finitely generated projective module $P$ over $R$ is a free module.