NAME:

STUDENT ID #:

You may use the textbook and your notes. Please work this exam on your own. Please show all your work. There are 5 problems for a total of 20 points.

1. (4 points) Let $G$ be a finite group of order $|G| < 60$. Show $G$ is solvable. (Guide: As groups of prime order are abelian, it suffices by induction to prove that every such $G$ has a non-trivial proper normal subgroup. Now consider the possible prime factorizations of the integer less than 60. those that are prime powers were treated in class ($p$-groups); those with only two prime factors are treated in the Elman notes; this leaves only one possible order to take care of.)

2. (8 points) Now prove that all groups of odd order less than or equal to 525 are solvable.

3. (4 points) Let $G$ be a finite group, $V$ a finite-dimensional complex vector space. Suppose $G$ acts on $V$ by linear automorphisms, that is, we have an action $G \times V \to V$ and for each $g \in G$ the permutation $\rho_g$ defined by $\rho_g(v) = gv$ is a linear transformation. Now assume that $W \subseteq V$ is a subspace such that for all $w \in W$ and $g \in G$ $gw \in W$ (we say that $W$ is an invariant subspace). Show that there is an inner product on $V$ such that the orthogonal complement to $W$ with respect to that inner product is also an invariant subspace.

4. (8 points) Let $G$ be a finite group. You have seen the commutator subgroup $G'$ in the homework exercises. Define inductively $G^{(1)} = G'$ and
$G^{(i+1)} = (G^{(i)})'$ for $i \geq 1$. Show: $G$ is solvable if and only if there exists an integer $n$ such that $G^{(n)} = \{e_G\}$.

5. (8 points) Let $G$ be a group. Define inductively normal subgroups $Z_i$ as follows. $Z_1 = Z(G)$ the center. If $Z_i(G)$ has been defined, let $\pi_i : G \to G/Z_i$ be the canonical homomorphism, and set $Z_{i+1} = \pi_i^{-1}(Z(G/Z_i))$ to be the pre-image of the center of $G/Z_i$.

(a) Show that this construction makes sense, that is, prove that $Z_i$ is actually normal in $G$ for all $i$.
(b) Show that $Z_i < Z_{i+1}$ for all $i$.
(c) Suppose there is an integer $n$ such that $Z_n = G$. Show that in this case, $G$ is solvable.
(d) Is the converse to (c) true? If so, prove it. If not, give a counterexample.