Problem 1. Do Exercise 1, Section 5.2, parts (a) – (g). Justify each answer.

Problem 2. For each of the following matrices $A \in M_{n \times n}(\mathbb{R})$, determine if $A$ is diagonalizable. If $A$ is diagonalizable, find an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$.

1. \[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix},
\]
2. \[
\begin{pmatrix}
1 & 3 \\
3 & 1
\end{pmatrix},
\]
3. \[
\begin{pmatrix}
1 & 4 \\
3 & 2
\end{pmatrix},
\]
4. \[
\begin{pmatrix}
7 & -4 & 0 \\
8 & -5 & 0 \\
6 & -6 & 3
\end{pmatrix}.
\]

Problem 3. For each of the following linear operators $T$ on a vector space $V$, determine if $T$ is diagonalizable. If $T$ is diagonalizable, find a basis $\beta$ for $V$ such that $[T]_{\beta}$ is a diagonal matrix.

1. $V = \mathbb{R}^3$ and $T$ is defined by $T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}$.
2. $V = P_2(\mathbb{R})$ and $T$ is defined by $T(ax^2 + bx + c) = cx^2 + bx + a$.
3. $V = P_3(\mathbb{R})$ and $T$ is defined by $T(f(x)) = f'(x) + f''(x)$ (where $f'(x)$ and $f''(x)$ are the 1st and the 2nd derivatives of $f(x)$, respectively).
4. $V = M_{2 \times 2}(\mathbb{R})$ and $T$ is defined by $T(A) = A^t$.

Problem 4. For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2 \times 2}(\mathbb{R})$, find $A^{1000}$.

(Hint: reduce the problem to raising a diagonal matrix to the 1000th power).

Problem 5. Suppose that $A \in M_{n \times n}(F)$ has two distinct eigenvalues, $\lambda_1$ and $\lambda_2$, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that $A$ is diagonalizable.

Problem 6. Prove that the eigenvalues of an upper triangular matrix $M$ are the diagonal entries of $M$.

Problem 7. Let $T$ be an invertible linear operator on a vector space $V$.

1. Prove that a scalar $\lambda \in F$ is an eigenvalue of $T$ if and only if $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.
2. Prove that the eigenspace of $T$ corresponding to $\lambda$ is the same as the eigenspace of $T^{-1}$ corresponding to $\lambda^{-1}$.
3. Prove that if $T$ is diagonalizable, then $T^{-1}$ is also diagonalizable.
Problem 8. Let $A \in M_{n \times n}(F)$.

(1) Prove that $A$ and $A^t$ have the same characteristic polynomial.

(2) It follows from (1) that $A$ and $A^t$ share the same eigenvalues with the same multiplicities. For any eigenvalue $\lambda$ of $A$ and $A^t$, let $E_\lambda$ and $E'_\lambda$ denote the corresponding eigenspaces for $A$ and $A^t$, respectively. Prove that for any eigenvalue $\lambda$, $\dim(E_\lambda) = \dim(E'_\lambda)$.

(3) Prove that if $A$ is diagonalizable, then $A^t$ is also diagonalizable.