Problem 1. Do Exercise 1, Section 2.4. Justify each answer.

Problem 2.

1. Which of the following pairs of vector spaces are isomorphic? Justify your answers.
   (a) \( F^3 \) and \( P_3(F) \).
   (b) \( F^4 \) and \( P_3(F) \).
   (c) \( M_{2 \times 2} (\mathbb{R}) \) and \( P_3(\mathbb{R}) \).
   (d) \( V = \{ A \in M_{2 \times 2} (\mathbb{R}) : \text{tr} (A) = 0 \} \) and \( \mathbb{R}^4 \).

2. Let \( V = \left\{ \begin{pmatrix} a & a + b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\} \). Construct an isomorphism from \( V \) to \( F^3 \).

Problem 3. Do Exercise 1, Section 2.5. Justify each answer.

Problem 4. For each of the following pairs of ordered bases \( \beta \) and \( \beta' \) for \( V \), find the change of coordinates matrix that changes \( \beta' \)-coordinates into \( \beta \)-coordinates.

1. \( \beta = \{ e_1, e_2 \} \) and \( \beta' = \{(a_1, a_2), (b_1, b_2)\} \), in \( V = \mathbb{R}^2 \).
2. \( \beta = \{(−1, 3), (2, −1)\} \) and \( \beta' = \{(0, 10), (5, 0)\} \), in \( V = \mathbb{R}^2 \).
3. \( \beta = \{(2, 5), (−1, −3)\} \) and \( \beta' = \{e_1, e_2\} \), in \( V = \mathbb{R}^2 \).
4. \( \beta = \{1, x, x^2\} \) and \( \beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\} \), in \( V = P_2(\mathbb{R}) \).

Problem 5. Given two matrices \( A, B \in M_{n \times n}(F) \), we say that \( B \) is similar to \( A \) if there exists an invertible matrix \( Q \) such that \( B = Q^{-1}AQ \). Similarity is an equivalence relation.

Recall that the trace of a matrix \( A \in M_{n \times n}(F) \) is the sum of its diagonal entries, that is \( \text{tr} (A) = A_{1,1} + A_{2,2} + \ldots + A_{n,n} \). Prove the following statements:

1. For any \( A, B \in M_{n \times n}(F) \), \( \text{tr} (AB) = \text{tr} (BA) \).
2. If \( A \) and \( B \) are similar, then \( \text{tr} (A) = \text{tr} (B) \).
3. Let \( V \) be a vector space with \( \dim(V) = n \), and let \( \beta, \beta' \) be two ordered bases for \( V \), and let \( T \in \mathcal{L}(V) \) be arbitrary. Then \( \text{tr} \left( [T]_{\beta} \right) = \text{tr} \left( [T]_{\beta'} \right) \).

Problem 6. Prove the following generalization of Theorem 2.23.

Let \( T : V \to W \) be a linear transformation, \( \dim(V), \dim(W) < \infty \). Let \( \beta, \beta' \) be ordered bases for \( V \), and let \( \gamma, \gamma' \) be ordered bases for \( W \). Then

\[
[T]_{\beta'}^\gamma = P^{-1} [T]_{\beta}^\gamma Q,
\]

where \( Q \) is the matrix that changes \( \beta' \)-coordinates into \( \beta \)-coordinates, and \( P \) is the matrix that changes \( \gamma' \)-coordinates into \( \gamma \)-coordinates.
Problem 7. Compute the determinants of the following matrices (and provide the details of your calculations).

(1) \[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 5 \\
-1 & 3 & 0
\end{pmatrix},
\]

(2) \[
\begin{pmatrix}
0 & 2 & 1 & 3 \\
1 & 0 & -2 & 2 \\
3 & -1 & 0 & 1 \\
-1 & 1 & 2 & 0
\end{pmatrix},
\]

(3) \[
\begin{pmatrix}
14 & 80 & -14 & -76 & -4 \\
0 & 2 & 1 & 3 & 0 \\
1 & 0 & -2 & 2 & 0 \\
3 & -1 & 0 & 1 & 0 \\
-1 & 1 & 2 & 0 & 0
\end{pmatrix}.
\]

Problem 8. Prove that \(\det(cA) = c^n \det(A)\) for any \(A \in M_{n \times n}(F)\) and \(c \in F\).