Problem 1. Show that the vectors $(1, 1, 0), (1, 0, 1)$ and $(0, 1, 1)$ generate $\mathbb{R}^3$.

Problem 2. Show that a subset $W$ of a vector space $V$ is a subspace if and only if $\text{Span}(W) = W$.

Problem 3. Let $S_1$ and $S_2$ be subsets of a vector space $V$ such that $S_1 \subseteq S_2$.

1. Show that then $\text{Span}(S_1) \subseteq \text{Span}(S_2)$.
2. If $\text{Span}(S_1) = V$, deduce that $\text{Span}(S_2) = V$.

Problem 4. Let $M_{m \times n}(\mathbb{R})$ be the vector space of all $m$-by-$n$ matrices with real entries.

For an $m \times n$ matrix $A \in M_{m \times n}(\mathbb{R})$, its transpose $A^t$ is the $n \times m$ matrix obtained from $A$ by interchanging the rows with the columns. That is, $(A^t)_{ij} = A_{ji}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. So for example if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

A symmetric matrix is a matrix $A$ such that $A^t = A$ (so it has to be a square matrix, that is $m = n$).

Let $W$ be the set of all symmetric matrices in $M_{2 \times 2}(\mathbb{R})$.

1. Show that $W$ is a subspace of $M_{2 \times 2}(\mathbb{R})$ (Hint: you will need to prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{2 \times 2}(\mathbb{R})$ and $a, b \in \mathbb{R}$).
2. Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
   Show that $\text{Span}\{(A_1, A_2, A_3)\} = W$.

Problem 5. Consider the following sets of vectors.

1. $\{(−1, 1, 2), (1, −2, 1), (1, 1, 4)\}$ in $\mathbb{R}^3$,
2. $\{(1, −1, 2), (2, 0, 1), (−1, 2, −1)\}$ in $\mathbb{R}^3$,
3. $\left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \\ -8 \end{pmatrix} \right\}$ in $M_{2 \times 2}(\mathbb{R})$.

Determine if they are linearly dependent or linearly independent (and justify).

Problem 6. Let $V = \mathbb{R}^3$. Find three vectors $w, v, z \in V$ with the following properties:

1. $\text{Span}\{\{w, v\}\} = \text{Span}\{\{v, z\}\} = \text{Span}\{\{w, v, z\}\}$,
2. $\text{Span}\{\{w, v, z\}\} \neq \text{Span}\{w, z\}$.

Suppose that $w, v, z$ are any three vectors with the above listed properties. Prove or disprove the following statements:

1. $w, v$ are linearly independent.
2. $v, z$ are linearly independent.
3. $w, z$ are linearly independent.
Problem 7. Determine whether the vectors 
\[ f(x) = \sin^2 x, \quad g(x) = \cos^2 x, \quad h(x) = 1 \]
in the vector space \( \mathcal{F}(\mathbb{R}, \mathbb{R}) \) of all functions from \( \mathbb{R} \) to \( \mathbb{R} \) are linearly independent.

Problem 8. Give three different bases for each of the following spaces:

1. \( \mathbb{R}^2 \)
2. \( M_{2 \times 2}(\mathbb{R}) \)
3. \( P_2(\mathbb{R}) \)