Final Exam Solutions
3/17/03

1. Calculate

\[ d_1 = \frac{1}{2} \left[ \log \left( \frac{100}{92} \right) + \left( 0.05 + 0.4 \right)^2 \right] \]
\[ = 5 \times \left( 0.105 + 0.07 \right)^2 = 5 \times 0.175^2 = 0.175^2 = 0.875 \]
\[ d_2 = d_1 - 0.2 = 0.675 \]

\[ N(d_1) = N(0.875) = 0.809 \]
\[ N(d_2) = N(0.675) = 0.750 \]

(a) \[ c = 100 N(d_1) - \left( \frac{90}{1.05} \right) N(d_2) \]
\[ = 100 \times 0.809 - 94.25 
\[ = 8.09 \]

(b) \[ \Delta = N(d_1) = 0.809 \]

2. (a) Consider portfolio \( \Pi \) consisting of forward agreement plus cash \( K e^{-r(T-t)} S_0 + th + \)

\[ \Pi(t) = F(t) + Ke^{-r(T-t)} \]

At time \( t=T \), this is \( F(t) + K \). Suppose \( K \) is the price of the stock. The forward agreement takes away the cash \( K \) and gives one share of stock \( S(T) \). Since \( \Pi(T) = S(T) \), the buy no-arbitrage \( \Pi(t) = S(t) \) for all \( t \), i.e., \( F(t) + Ke^{-r(T-t)} = S(t) \)

or

\[ F(t) = S(t) - Ke^{-r(T-t)} \]
(b) \( F_t = -r \text{Ke}^{r(T-x)} \)
\[
F_s = 1 \\
F_{Ss} = 0 \\
S_0 = \frac{F_t + \frac{1}{2} \sigma^2 S^2 F_{ss} + r(SF_s - E)}{r} = r \text{Ke}^{r(T-x)} + r(S - (S - \text{Ke}^{r(T-x)})) = 0
\]
which shows that \( F \) satisfies the Black-Scholes PDE.

3. (a) At \( t=T \), \( d_e + dp = 1 \). Since this is risk free, then it discounts backwards in time like cash, i.e.,
\[
d_e + dp = e^{-r(T-t)}
\]

(b) For the CRR
\[
\rho^* = \frac{e^{0.25} - 0.9}{1 - 0.2} = 0.25 \times \frac{1.25}{0.8} = \frac{1.25}{0.8} S_0 = 1.21 S_0
\]
\[
q^* = 1.15 S_0
\]
\[
p^* = 0.81 S_0
\]
\[
q^* = 1.375
\]
For \( K = S_0 \), \( d_e \) [CRR] has payout only for \( S_e = u^2 S_0 \) and \( dp \) has payout for the other 2 outcomes.

\[
d_e = e^{-r \cdot dt} E^* \left[ d_e(T) \right] = e^{-0.05 \cdot \rho^*} = 0.95 \times (0.625)^2 = 0.372
\]
\[ \begin{align*}
  d_p &= e^{-2rdt} E^* \left[ d_p(t) \right] \\
  &= e^{-2rdt} \left( 2p^* q^* + q^* q \right) \\
  &= e^{-2rdt} (1 - q^* q) \\
  &= e^{-0.05} (1 - (0.25)^2) \\
  &= 0.95 (1 - 0.391) \\
  &= 0.578
\end{align*} \]

(c) Check that for \( t = 0 \),

\[ d_p + d_c = e^{-rT} \]

i.e.,

\[ 0.372 + 0.578 = e^{-0.05} = 0.95 \] \( \checkmark \)
4. Exercise if \( S_0 > K \)

(a) Real probability of exercise
\[
\begin{align*}
\rho_e &= \begin{cases} 
0 & \text{if } K > u^2 S_0 \\
\rho & \text{if } u d S_0 < K < d^2 S_0 \\
1 - q^2 & \text{if } d^2 S_0 < K < u d S_0 \\
1 & \text{if } K < d^2 S_0
\end{cases}
\end{align*}
\]

(b) \( \rho_e^* \)
\[
\begin{align*}
\rho_e^* &= \begin{cases} 
0 & \text{if } K > u^2 S_0 \\
\rho^* & \text{if } u d S_0 < K < d^2 S_0 \\
1 - q^* & \text{if } d^2 S_0 < K < u d S_0 \\
1 & \text{if } K < d^2 S_0
\end{cases}
\end{align*}
\]

(c) If there's a risk premium, then \( p > p^* \) and \( q < q^* \)
\[
\begin{align*}
S_0 &> p^2 \\
1 - q^2 &> 1 - q^*
\end{align*}
\]
\[\Rightarrow \rho_e > \rho_e^* \text{ in all 4 cases}\]

(d) For three step model
\[
\rho_e = \begin{cases} 
0 & \\
\rho^3 & \\
p^3 + 3 p^2 q = p^3 + 3 p^2 (1-p) = 3 p^2 - 2 p^3 & \\
1 - q^3 & \\
1
\end{cases}
\]

As before \( p^3 > p^*_3 \), \( 1 - q^3 > 1 - q^*_3 \)
Finally, the function $f(p) = 3p^2 - 2p^3$ has derivative
$f'(p) = 6p - 6p^2 > 0$ for $0 < p < 1$. So $f(p) \geq f(p^*)$ since $p > p^*$. So

$$3p^2 - 2p^3 \geq 3p^* - 2p^3.$$