Midterm Solutions

1. (i) Since $f$ is an option price, it satisfies the Black-Scholes equation

$$-f_t = \frac{1}{2}\sigma^2 S^2 f_{ss} + rSf_s - rf.$$  

It’s “initial condition” is its value at the expiration time $t = T$. At this time the holder pays an amount $X$ and receives one share of the stock which is worth $S_T$, so that

$$f(S_T, T) = S_T - X.$$  \hspace{1cm} (1)

The formula $f(S, t) = S - X e^{-r(T-t)}$ satisfies

$$f_t = -rX e^{-r(T-t)} = r(f - S)$$

$$f_S = 1$$

$$f_{SS} = 0$$

Therefore

$$\frac{1}{2}\sigma^2 S^2 f_{ss} + rSf_s - rf = rS - rf$$

which shows that $f$ satisfies the Black-Scholes equations, and

$$f(S, T) = S - X$$

which shows that $f$ satisfies the initial condition.

(ii) At expiration, $f = S - X$ whereas the value of the call is $c = \max (S - X, 0)$. In particular, $f \leq c$ at $t = T$. By no-arbitrage, it follows that $f \leq c$ for all $t$. In addition, since $f < c$ with some positive probability, then in fact $f < c$ for all $t < T$.

2. (i) By its definition

$$x_5 = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5.$$  \hspace{1cm} (2)

Since these are each independent Gaussians with variance 1, then

$$\sigma^2(x_5) = \sigma^2(\omega_1) + \sigma^2(\omega_2) + \sigma^2(\omega_3) + \sigma^2(\omega_4) + \sigma^2(\omega_5)$$

$$= 5.$$  \hspace{1cm} (3)
(ii) Since $x_5$ is a Gaussian random variable with variance 5 and mean 0, then its probability density is $p(x) = (10\pi)^{-1/2} e^{x^2/10}$. It follows that

$$E[\max (x_5, 0)] = \int_{-\infty}^{\infty} \max (x, 0) p(x) dx$$

$$= \int_{0}^{\infty} x p(x) dx$$

$$= \int_{0}^{\infty} x (10\pi)^{-1/2} e^{x^2/10} dx$$

$$= (10\pi)^{-1/2} \int_{0}^{\infty} \sqrt{5} y e^{-y^2/2} \sqrt{5} dy \quad \text{in which } x = \sqrt{5} y$$

$$= (5/2\pi)^{1/2} \int_{0}^{\infty} \frac{-d e^{-y^2/2}}{dy} dy$$

$$= (5/2\pi)^{1/2} \left[ -e^{-y^2/2} \right]_{y=0}^{y=\infty}$$

$$= (5/2\pi)^{1/2} \cdot (4)$$

3. The Black-Scholes formula for a put price with $t = 0$ is

$$p = X e^{-rT} N(-d_2) - S_0 N(-d_1)$$

$$d_1 = \frac{\log(S/X) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$ 

For the security in this problem

$$S_0 = X = 1, \ T = 4, \ r = .05, \ \sigma = .2$$

$$d_1 = \frac{0 + (.05 + .04/2)4}{.2 \sqrt{4}} = .7$$

$$d_2 = .7 - .4 = .3.$$ 

From the table on the back of the exam

$$N(-d_1) = N(-.7) = .24$$

$$N(-d_2) = N(-.3) = .38$$

$$e^{-rT} = e^{-2} = .82.$$ 

(5)
Insert this into the Black-Scholes formula to get the call price
\[
c = 0.82 \times 0.38 - 0.24 \\
= 0.072.
\]

4. From put call parity, we have
\[
p - c = S - X e^{-r(T-t)}.
\]
Therefore the equation \( p = x \) at \( S = \bar{S}(t) \) implies that
\[
\bar{S}(t) = X e^{-r(T-t)}.
\] (6)