1. The Black-Scholes equation is

$$-f_t = \frac{1}{2} \sigma^2 S^2 f_{SS} + rSf_S - rf$$

The value for a stock is \( f(S,t) = S \).

Then \( f_t = 0 \)

\( f_s = 1 \)

\( f_{ss} = 0 \)

So in B-S this gives

$$0 = \frac{1}{2} \sigma^2 S^2 \cdot 0 + rS \cdot 1 - rS$$

$$= 0 \quad \checkmark$$

The value for a cash account is \( f = A = A_0 e^{rt} \)

\( f_t = A_t = rA_0 e^{rt} = rA \)

\( f_s = f_{ss} = A_s = A_{ss} = 0 \)

So in B-S this gives

$$-rA = 0 + 0 - rA$$

2. Consider suppose that \( S_0 \) and \( R_0 \) are the values at \( t = 0 \). Form the portfolio

$$P = R_0 S - S_0 R$$

At \( t = 0 \)

$$P = P_0 = R_0 S_0 - S_0 R_0 = 0$$

At \( t = dt \), there are 4 possibilities
\[ P = (R_o (uS_o) - S_o (uR_o)) = 0 \quad \text{if } S \text{ and } R \text{ go up} \]
\[ = (R_o (dS_o) - S_o (dR_o)) \quad \text{if } S \text{ and } R \text{ go down} \]
\[ = (u-d) R_o S_o > 0 \quad \text{if } S \text{ goes up and } R \text{ goes down} \]
\[ = -(u-d) S_o R_o < 0 \quad \text{if } S \text{ goes down and } R \text{ goes up} \]

By assumption, the third possibility never happens, i.e., if \( S \) goes up, then \( R \) goes up. It follows that \( P \leq 0 \) at \( dt \). No-arbitrage then implies that \( P = 0 \) for all possibilities at \( dt \).

But then the 4th possibility does not happen. So, \( S \) and \( R \) either both go up or both go down. This implies that \( p' = q' \).
3. The risk neutral probability is

\[ p = \frac{e^{rd} - d}{u - d} = \frac{e^{r - .9} - 1.2 - .9}{1.2 - .9} = \frac{11 - .9}{1.2 - .9} = \frac{3}{3} \]

The payout \( S_u \) at \( T = 1.0 = dt \) is

\[ c_1 = \begin{cases} c_u & \text{up step} \\ c_d & \text{down step} \end{cases} \]

\[ c_u = \max(u S_0 - X, 0) \]

\[ c_d = \max(d S_0 - X, 0) \]

\[ = \begin{cases} 120 - 100, 0 \\ 90 - 100, 0 \end{cases} \]

\[ = 20 \]

\[ = 0 \]

\[ S_0 = e^{-rd + \frac{1}{2} \sigma^2 (p c_u + (1-p) c_d)} \]

\[ = \frac{1}{1.1} \cdot \frac{2}{3} \cdot 20 \]

\[ = \frac{40}{3.3} \]
4. By a telescoping sum
\[ x_2 = (x_2 - x_1) + (x_1 - x_0) + x_0 \]
\[ = \sqrt{dt} \omega_2 + \sqrt{dt} \omega_1 + x_0 \]
\[ = \int_0^t \omega dt + x_0 \]

with
\[ \sigma^2 = (\sqrt{dt})^2 + (\sqrt{dt})^2 = 2dt = 4st, \text{ i.e. } \sigma = 2\sqrt{st} \]
in which \( \omega \) is \( N(0,1) \)

Also
\[ y_4 = (y_4 - y_3) + (y_3 - y_2) + (y_2 - y_1) + (y_1 - y_0) + y_0 \]
\[ = \sqrt{st} y_4 + \sqrt{st} y_3 + \sqrt{st} y_2 + \sqrt{st} y_1 + x_0 \]
\[ = Y + x_0 \]

with
\[ \sigma^2 = (\sqrt{st})^2 + (\sqrt{st})^2 + (\sqrt{st})^2 + (\sqrt{st})^2 \]
\[ = 4st \]
\[ \text{i.e. } \tau = 2\sqrt{st} \]
and \( Y \) is \( N(0,1) \)

So \( x_2 \) and \( y_4 \) have the same statistics.