NOTATION. Use the following notation:

For an option, \( T \) is exercise time (in years) and \( K \) is strike price.

For the Black-Scholes model, \( S \) is equity price; \( t \) is time; \( \sigma \), \( \mu \) and \( r \) are the volatility, average growth rate and (continuously compounded) risk-free rate of return.

For the CRR model, \( S_0 \) is the initial stock price, \( S_n \) is the stock price after \( n \) steps, \( u \) and \( d \) denote the factors for increase and decrease of the equity price, \( dt \) is the time step, the real probabilities are \( p \) and \( q \), and the risk-neutral probabilities are \( p^* \) and \( q^* \). Use the continuously compounded interest rate \( r \) for the CRR model, as well as for the Black-Scholes model.

You may use the facts that \( \log(100/90) = .105 \) and \( \exp(.05) = 1.05 \).

1. Consider a call option, for an equity following the Black-Scholes model, with \( T = 1 \) and \( K = 90 \) on an equity with initial price \( S(0) = 100 \), and with \( \sigma = 0.2 \), \( \mu = 0.1 \) and \( r = 0.05 \).

(a) What is the value \( c(0) \) of the call option at \( t = 0 \)?

(b) What is the value of \( \Delta \) for this option at \( t = 0 \)?

2. Consider a forward contract to purchase an equity at strike price \( K \) and at time \( T \).

(a) Use a no arbitrage argument to show that the price of the forward contract is \( F(S, t) = S - Ke^{-r(T-t)} \).

(b) Show that \( F \) solves the Black-Scholes PDE.

3. Define a digital call \( d_c \) and a digital put \( d_p \) with strike price \( K \) and exercise time \( T \) to have payouts

\[
d_c(S, T) = \begin{cases} 1 & \text{if } S \geq K \\ 0 & \text{if } S < K \end{cases}
\]

\[
d_p(S, T) = \begin{cases} 0 & \text{if } S \geq K \\ 1 & \text{if } S < K \end{cases}
\]

(a) Use a no arbitrage argument to show that \( d_c + d_p = e^{-r(T-t)} \).

(b) Find the initial price of \( d_p \) and \( d_c \) on a two step CRR model with \( u = 1.1 \), \( d = 0.9 \), \( S_0 = K = 100 \), \( r = 0.05 \), \( dt = .5 \) and \( p = 0.5 \).

(c) Show that the result from (b) satisfies the “digital put-call parity” relation in (a).

4. Consider a call option for a two-step CRR model. Define the probability for exercise of the option to be \( p_c \) in the real world and \( p_c^* \) risk-neutral world.

(a) Find a formula for \( p_c \).

(b) Find a formula for \( p_c^* \).

(c) Assuming that the CRR model has a risk premium (corresponding to risk aversion), show that \( p_c^* < p_c \) for all values of the strike price \( K \).

(c) (Bonus Problem) For a three-step CRR model, show that the inequality in (c) for some of values of \( K \) and \( p \).