1. The Black-Scholes equation is

\[ f_t + \frac{1}{2} \sigma^2 S^2 f_{S^2} + r S f_S - r f \]

Look for

\[ f(S, t) = S^2 e^{\alpha(T-t)} \]

This satisfies the pay-off condition

\[ S^2 = f(S, T) \]

For \( t < T \),

\[ f_t = -\alpha S^2 e^{\alpha(T-t)} \quad f_T = -\alpha f \]

\[ f_S = 2 S e^{\alpha(T-t)} \quad S f_S = 2 f \]

\[ f_{S^2} = 2 S e^{\alpha(T-t)} \quad S^2 f_{S^2} = 2 f \]

Insert in Black-Scholes and cancel \( f \) to get

\[ 0 = -\alpha + \sigma^2 + 2r - r \]

\[ = -\alpha + \sigma^2 + r \]

\[ \alpha = \sigma^2 + r \]

2. If \( f(S, T) > g(S, T) \) for all \( S \), then by no-arbitrage

\[ f(S, t) > g(S, t) \] for all \( S, t \)
3 (a) For the binomial model, after 4 steps, \( S \) has the values

\[ 146, 120, 98, 80, 65.6 \]

The put payout is 0 for all but the last of these, corresponding to 4 down steps. For this it is \( 75 \cdot 66 = 9.4 \)

The risk neutral probability \( p \) (for up step) is

\[ p = \frac{e^{0.02/4} - 1.1}{1.1 - 0.9} = \frac{1.005 - 0.9}{0.2} = 0.25 \]

\( S_0 \)

\[ p_0 = e^{-0.02} E[\text{payout}] \]

\[ = \frac{e^{-0.02} (0.5)^4}{2} \]

\[ = 0.0534 \]

\( = 0.1069 \)

(b) \( p_0 = X e^{-r} N(-d_2) - S_0 N(-d_1) \)

\[ = 75 e^{-0.02} N(-d_2) - 100 N(-d_1) \]

\[ d_1 = \left( \log \left( \frac{100}{75} \right) + 0.02 + 0.16/2 \right)/0.4 \]

\[ = (1.287 + 0.1)/0.4 \]

\[ = 4.019 \]

\[ d_2 = d_1 - 4 = 0.69 \]

\[ N(-d_1) = 0.166 \]

\[ N(-d_2) = 0.2843 \]

\[ p_0 = 75 \cdot 0.98 \cdot 0.284 - 100 \cdot 0.166 = 2.08 - 16.6 \]

\[ = 0.42 \]
4. \( X_4 = 0 \) requires an equal number of up and down steps. The possibilities are:

\[
(+ + - -), (+ - + -), (+ - - +), (- - + +), (- + - +), (- + + -)
\]

This is 6 out of 16 equally likely paths, so the probability is

\[
p = \frac{6}{16} = \frac{3}{8}
\]