Price of Calls and Puts from Black-Scholes

The Black-Scholes equation for the price $f = f(t, S)$ of an option is

$$-f_t = \frac{1}{2}\sigma^2 S^2 f_{ss} + f S f_s - rf.$$  \hspace{1cm} (1)

In our stock price model, the value of $S$ can never become 0 or negative. So this equation is to be solved on $S > 0$.

This is a general equation that is valid for any option. How do we insert information about the specific option we wish to price? Through the “initial conditions”, as shown next.

Consider a call with price $f = c(S, t)$. The only information we have about $c$ is that

$$c(S, T) = \max(0, S - X)$$

at the expiration time $t = T$. We wish to find the price for $t < T$. The resulting equation is

$$-c_t = \frac{1}{2}\sigma^2 S^2 c_{ss} + rSc_s - rc \quad \text{for } t < T \hspace{1cm} (2)$$

$$c = \max(0, S - X) \quad \text{for } t = T. \hspace{1cm} (3)$$

As stated above, this is to be solved for $S > 0$.

We call the condition (3) an initial condition, in analogy to the use of initial data for differential equations. A better name might be a “final condition”. It is fortunate that the data is specified at the end of the time period $t \leq T$, because this is consistent with the $-f_t$ term in (2). That is, the natural direction in (2) is backwards in time. So it makes sense to give data at the end and solve backwards.

Black and Scholes succeeded in solving (2) and (3). The solution is

$$c(S, t) = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$  \hspace{1cm} (4)

in which
\[
d_1 = \frac{\log(S/X) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \\
d_2 = \frac{\log(S/X) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}
\]

and \( N(x) \) is the cumulative distribution function for an \( N(0, 1) \) random variable. \( N \) satisfies

\[
N(x) = (2\pi)^{-\frac{1}{2}} e^{-x^2/2} \\
N \to 0 \quad \text{as} \quad x \to -\infty
\]

so that

\[
N(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-x'^2/2} \, dx' = \frac{1}{2} + \frac{1}{2} \operatorname{erf}(x/\sqrt{2})
\]

in which \( \operatorname{erf}(y) \) is the “error function” defined by

\[
\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_{0}^{y} e^{-t^2} \, dt.
\]

This is a standard function that appears in many mathematical handbooks.

The Black-Scholes formula (4) provides the price \( c(S,t) \) for a call as a function of the current time \( t \) and the current stock price \( S \). This formula can be written in the following way

\[
c(S,t) = e^{-r(T-t)} \{ SN(d_1)e^{r(T-t)} - XN(d_2) \}.
\]

The term \( N(d_2) \) is the probability of exercise of the option, in the “risk-neutral world” that will be discussed in the next lecture. So \( XN(d_2) \) is the strike price times the probability that it is paid.

The term \( SN(d_1)e^{r(T-t)} \) is the expected value at \( t = T \) of a variable that is \( S(T) \) if \( S(T) > X \) and 0 otherwise. The factor \( e^{-r(T-t)} \) is the discount factor.

There is a similar formula for a put. The price \( p \) of a put satisfies

\[-p_t = \frac{1}{2} \sigma^2 S^2 p_{ss} + r S p_s - rp \quad \text{for} \quad t < T \\
p = \max(X - S, 0) \quad \text{for} \quad t = T.
\]

The solution is

\[
p = Xe^{-r(T-t)}N(-d_2) - SN(-d_1).
\]