Do the following problems. Show all your work. Partial credit will be given only for exhibited work. Good Luck!

1. (10 points). Derive the formula for the risk neutral measure on a one step binomial tree, explaining carefully any ideas you use.
2. (25 points) A stock currently costs $100 per share. Over each of the next two three-month periods it is expected to increase or decrease by 10%. Consider a six month European put. Suppose the strike price $K$ of the put is $95 and the risk-free interest rate is 8% per annum.

(a) (7 points) Compute the risk neutral measure for the associated binomial tree. Use this computation to answer the following questions.

(b) (5 points) What is the value of the option at $t = 0$?

(c) (5 points) Suppose the option is a call instead. What is its value at $t = 0$?

(d) (5 points) What is the implied volatility (per annum) using the Cox, Ross, Rubinstein scheme?

(e) (3 points) Suppose the put is American. What is its value and why?
3. (12 points)

(a) (6 points) What are the two characteristic properties of a Wiener process?

(b) (6 points) A quantity $Q$ obeys the law $dQ = 3dt + 4dx$, where $dx$ is a Wiener process, for two years. Then it obeys the law $dQ = 2dt + 3dx$ for 3 years. What is the expected value of $Q$ after 5 years? What is the standard deviation of the quantity after 5 years?
4. (30 points)

(a) (10 points) A stock $S$ follows the process $dS = \mu S dt + \sigma S dz$, where $a$ and $b$ are non-negative constants. What is the name of such a process? What are the names of $\mu$ and $\sigma$?

(b) (10 points) Let $X = \ln(S)$ be a new process. Derive, using Ito’s Lemma, the process followed by $X$. What is the name of this type of process?

(c) (10 points) A stock has an initial price of $40, an expected return of 10% per annum and a volatility of 20% per annum. Give explicitly the probability distribution of $X$ after 6 months have elapsed. Give a 95% confidence interval, centered on the the expected value, for $X$ at this time, and hence, exponentiating, one for $S$. 
5. (23 points)

(a) (5 points) What property of the Black-Scholes differential equation enables us to use risk-neutral valuation to find solutions to it?

(b) (5 points) What is the importance of using \( \frac{\partial f}{\partial S} \) in the portfolio \( \Pi = -f + \frac{\partial f}{\partial S} S \), i.e. what is special about the portfolio? Here \( f \) is the price of one unit of derivative (which is hence shorted in the portfolio \( \Pi \).

(c) (13 points) Use Black-Scholes to price a call option has a strike price of 30 on a stock with current price of 28. Suppose that the volatility of the stock is 20% per annum and that the risk free rate is 7% per annum. Suppose that the option expires in 3 months.