DEFINABLE PRINCIPAL SUBCONGRUENCES

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Abstract. For varieties of algebras, we present the property of having "definable principal subcongruences" (DPSC), generalizing the concept of having definable principal congruences. It is shown that if a locally finite variety $V$ of finite type has DPSC, then $V$ has a finite equational basis if and only if its class of subdirectly irreducible members is finitely axiomatizable. As an application, we prove that if $A$ is a finite algebra of finite type whose variety $V(A)$ is congruence distributive, then $V(A)$ has DPSC. Thus we obtain a new proof of the finite basis theorem for such varieties. In contrast, it is shown that the group variety $V(S_3)$ does not have DPSC.

1. Introduction

We consider only varieties of finite type. Following Baldwin and Berman [3] and McKenzie [10], let us say that a first-order formula $\Gamma(u, v, x, y)$ is a congruence formula if it is positive existential and $\Gamma(u, v, x, x) \rightarrow u \approx v$ holds in all algebras of the relevant type. It follows that $\Gamma(u, v, x, y)$ implies \( \langle u, v \rangle \in Cg(x, y) \) (the principal congruence relation generated by identifying $x$ and $y$) in any algebra of the type. A typical congruence formula expresses the assertion that $\langle u, v \rangle$ can be reached from $\langle x, y \rangle$ by using one of finitely many Mal'tsev congruence schemes [7].

For some congruence formulas $\Gamma$ and instances of $x, y$ in an algebra, it is the case that $\Gamma(\underline{x}, y)$ is $Cg(x, y)$. A useful observation [10] is that this case can be described by a first-order formula $\Pi_{\Gamma}(x, y)$; specifically, $\Pi_{\Gamma}(x, y)$ asserts that $\Gamma(\underline{x}, y)$ is an equivalence relation compatible with the (finitely many) basic operations and also that $\Gamma(x, y, x, y)$ holds.

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A variety $V$ is said to have **definable principal congruences** (DPC) [3] if there is a first-order formula $\Gamma(u, v, x, y)$ such that in any $B \in V$, $(c, d) \in Cg(a, b)$ if and only if $B \models \Gamma(c, d, a, b)$. If $V$ does have DPC, then $\Gamma$ can be taken to be a congruence formula.

McKenzie [10] proves that if $V$ is a variety of finite type with DPC and only finitely many subdirectly irreducible members up to isomorphism, all finite, then $V$ is finitely based. We generalize this fact by defining the concept of having definable principal subcongruences (DPSC) and showing (Theorem 1) that if $V$ is a locally finite variety of finite type with DPSC for which the class of subdirectly irreducible members is definable (finitely axiomatizable), then $V$ is finitely based. An application is to congruence distributive varieties generated by a finite algebra $A$ of finite type, which are shown to have DPSC (Theorem 2). The resulting proof of the finite basis theorem [1, 9] for this congruence distributive case avoids dependence on computation with Jónsson terms [8]; cf. [1, 9, 2].


2. **DEFINABLE PRINCIPAL SUBCONGRUENCES**

**Definition.** A variety $V$ has **definable principal subcongruences** (DPSC) if there are congruence formulas $\Gamma_1(u, v, x, y)$ and $\Gamma_2(u, v, x, y)$ such that given any algebra $B \in V$ and elements $a, b \in B$ with $a \neq b$ there exist elements $c, d \in B$ with $c \neq d$ for which $B \models \Gamma_1(c, d, a, b)$ and $B \models \Pi_{\Gamma_2}(c, d)$.

In essence, the condition for DPC says that the variety has a finite list of congruence schemes [7] sufficient to compute any principal congruence, while the condition for DPSC says that the variety has a finite list of congruence schemes sufficient to reach a principal congruence that can be fully computed by a predetermined finite list of congruence schemes. Observe that DPC implies DPSC.

An instructive example is the variety $V(M_3)$, where $M_3$ is the five-element modular lattice with three atoms. By Theorem 2 below, $V(M_3)$ has DPSC, but McKenzie [10] shows that $V(M_3)$ does not have DPC. McKenzie observes that $V(M_3)$ contains lattices $P_n$ for $n = 1, 2, \ldots$, of which $P_4$ is shown in Figure 1. The computation $(b, 1) \in CgP_4(a, b)$ requires a sequence of transitivities of length at least $n$, so there cannot be a single formula for principal congruences and DPC fails. On the other hand, the condition for DPSC is satisfied; for example, in $P_4$ with $a, b$ as indicated, one can choose $c, d$ as shown and then a typical pair $(r, s) \in Cg(c, d)$ is reached via a computation whose complexity has a bound depending only on the variety. See also [4].
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Figure 1. The lattice $P_4$ of McKenzie

A class of similar algebras is said to be \textit{finitely axiomatizable} (or \textit{strictly elementary} or \textit{definable}) if it is the class of models of some first-order sentence. By the compactness theorem, the finitely axiomatizable varieties are simply those that are finitely based.

As mentioned, McKenzie [10] showed that a variety of finite type with DPC and with only finitely many subdirectly irreducible members, all finite, is finitely based. The following fact is a generalization. For any class $\mathcal{K}$ of similar algebras, let $\mathcal{K}_S$ denote the class of subdirectly irreducible members of $\mathcal{K}$.

**Theorem 1.** A variety $V$ with definable principal subcongruences is finitely based if and only if $V_S$ is finitely axiomatizable.

The proof depends on the following lemma. For convenience, let us say that a class $\mathcal{K}$ of similar algebras has a property “doubly” if both $\mathcal{K}$ and $\mathcal{K}_S$ have the property.

**Lemma.** If a variety $V$ is contained in a doubly finitely axiomatizable class $\mathcal{K}$, then $V$ is either doubly finitely axiomatizable or doubly not finitely axiomatizable.

**Proof** (after Jónsson [9]): First suppose that $V$ is not finitely axiomatizable. Then there exists an index set $I$, algebras $A_i \not\in V$, $i \in I$, and an ultrafilter $\mathcal{U}$ on $I$ such that the resulting ultraproduct $A^*$ is in $V$, by [6] Theorem 4.1.12, or by taking $I = \omega$ and for each $i$ choosing...
$A_i$ to satisfy all $i$-variable laws of $V$ but not all laws. If we replace each $A_i$ by one of its subdirectly irreducible subdirect factors not in $V$, then $A^*$ is replaced by a homomorphic image, so without loss of generality we may assume that each $A_i$ is subdirectly irreducible. Further, since $A^* \in \mathcal{K}$, which is finitely axiomatizable, we have $\{i \in I : A_i \in \mathcal{K}\} \subseteq \mathcal{U}$, so without loss of generality we may assume $A_i \in \mathcal{K}$ for all $i$. Then $A_i \in \mathcal{K}_S$ for all $i$, and since $\mathcal{K}_S$ is axiomatizable, $A^*$ is subdirectly irreducible. Thus $A_i \notin V_S$ for all $i$ but $A^* \in V_S$. Therefore $V_S$ is not finitely axiomatizable.

Suppose on the other hand that $V$ is finitely axiomatizable. Then so is $V_S = V \cap \mathcal{K}_S$. □

**Proof** of Theorem 1: Let $\Gamma_1$ and $\Gamma_2$ be congruence formulas witnessing DPSC for $V$ and let $\mathcal{K}$ be the class of all algebras (of the type of $V$) for which $\Gamma_1$ and $\Gamma_2$ witness DPSC. Observe that $\mathcal{K}$ is the class of models of

$$\Phi \equiv (\forall a, b)[a \neq b \rightarrow (\exists c, d)[c \neq d \land \Gamma_1(c, d, a, b) \land \Pi_{\Gamma_2}(c, d)]]$$

while $\mathcal{K}_S$ is the class of models of $\Phi \land \Psi$ for

$$\Psi \equiv (\exists r, s)[r \neq s \land (\forall a, b)[a \neq b \rightarrow (\exists c, d)[\Gamma_1(c, d, a, b) \land \Gamma_2(r, s, c, d)]]].$$

Since $V \subseteq \mathcal{K}$, the Lemma applies. □

**Remarks.** The same kind of argument would apply if it is the class of finitely subdirectly irreducible members of $V$ that is finitely axiomatizable.

3. **Congruence-distributive varieties generated by a finite algebra**

**Theorem 2.** Let $A$ be a finite algebra of finite type for which $V(A)$ is congruence distributive. Then $V(A)$ has definable principal subcongruences.

The proof depends on this fact about embeddings in a product:

**Observation.** In a congruence distributive variety, consider an embedding $C \hookrightarrow \prod_{i \in I} A_i$, where $C$ is finite. Let $p, q, r, s \in C$. Then $\langle r, s \rangle \in \text{Cg}^C(p, q)$ in $C$ if and only if the same holds in the projected image of $C$ in each factor, i.e., for each $i \in I$ we have $\langle \bar{r}_i, \bar{s}_i \rangle \in \text{Cg}^{\pi_i(C)}(\bar{p}_i, \bar{q}_i)$, where $\bar{r}_i, \bar{s}_i, \bar{p}_i, \bar{q}_i$ are the images in $A_i$.

Indeed, “only if” is automatic. For “if”, observe that $\text{Cg}^C(r, s) \leq \text{Cg}^C(p, q) \land \ker \pi_i$ for each $i$. Since $C$ is finite there are only finitely many possible kernels, so that the distributive law applies: $\text{Cg}^C(r, s) \leq$
\[ \cap_{i \in I} (C_i^G(p, q) \vee \ker \pi_i) = C_G(p, q) \vee (\cap_{i \in I} \ker \pi_i) = C_G(p, q) \vee 0 = C_G(p, q). \]

**Proof of Theorem 2:** By Jónsson’s Lemma [8], \( V(A) \) has up to isomorphism only finitely many subdirectly irreducible members, all finite. Let \( N \) be the maximum of their cardinalities. We proceed as follows. Given any algebra \( B \in V(A) \) and \( a \neq b \) in \( B \), we shall construct a subalgebra \( D \) of \( B \) with at most \( N \) generators, including \( a \) and \( b \), and designate \( c \neq d \) in \( D \) with \( C_G^D(c, d) \leq C_G^D(a, b) \). Next, given any \( r, s \in B \) with \( C_G^B(r, s) \leq C_G^B(c, d) \), we shall let \( C \) be the subalgebra of \( B \) generated by \( D \) and \( r, s \) and show that \( \langle r, s \rangle \in C_G^C(c, d) \). By local finiteness, \( |D| \) and \( |C| \) have finite bounds depending only on \( A \). Therefore there are congruence formulas \( \Gamma_1(u, v, x, y) \) and \( \Gamma_2(u, v, x, y) \), depending only on \( A \), with \( \Gamma_1(c, d, a, b) \) holding in \( D \) and hence in \( B \), and with \( \Gamma_2(r, s, c, d) \) holding in \( C \) and hence in \( B \), as required. Thus \( V(A) \) has DPSC.

To construct \( D \), let \( B \leftrightarrow \prod_{i \in I} S_i \) be a subdirect representation of \( B \), with coordinate maps \( \pi_i : B \to S_i, i \in I \). Choose \( j \in I \) so that \( n(j) = |S_j| \) is as large as possible subject to \( \pi_j(a) \neq \pi_j(b) \). Choose preimages \( e_1, \ldots, e_{n(j)} \in B \) of the elements of \( S_j \) under \( \pi_j \), with \( e_1 = a \) and \( e_2 = b \). Let \( D \) be the subalgebra of \( B \) generated by \( e_1, \ldots, e_{n(j)} \). Thus \( \pi_j(D) = S_j \). For convenience, write \( \pi_D^j \) for \( \pi_j|_D \).

Since \( S_j \) is subdirectly irreducible, \( \ker \pi_D^j \) is completely meet irreducible in \( \text{Con}(D) \). By the congruence distributivity of \( V(A) \), the interval \([0, \ker \pi_D^j] \) in \( \text{Con}(D) \) is a prime ideal; therefore its complement is a dual ideal whose least element \( \alpha \) is join-prime. In particular, \( \alpha \) is the least congruence on \( D \) not under \( \ker \pi_D^j \). But \( C_G^D(a, b) \nleq \ker \pi_D^j \) we have \( \alpha \leq C_G^D(a, b) \). Moreover, since \( \alpha \) is join-prime and is a finite join of principal congruences, \( \alpha \) is principal, say \( \alpha = C_G^D(c, d) \).

Let us say that say \( i \) splits \( u, v \in B \) if \( \pi_i(u) \neq \pi_i(v) \). Observe that if \( i \) splits \( c, d \), then \( C_G^D(c, d) \nleq \ker \pi_i \) and \( i \) also splits \( a, b \), so by the minimality of \( \alpha = C_G^D(c, d) \) we have \( \ker \pi_i \leq \ker \pi_D^j \). Then there is an induced map of \( D/\ker \pi_D^j \cong \pi_D(D) \) onto \( D/\ker \pi_D^j \cong \pi_D(D) = S_j \).

By the choice of \( j \), \( \pi_i \) maps \( D \) onto \( S_i \).

Now let \( r, s \in B \) be given with \( C_G^B(r, s) \leq C_G^B(c, d) \). As mentioned, let \( C \) be the subalgebra of \( B \) generated by \( D \) and \( r, s \). Again by the local finiteness of \( V(A) \), \( C \) is finite. We apply the Observation to \( c, d, r, s \) and \( C \leftrightarrow \prod_{i \in I} S_i \), as follows. If \( i \) splits \( c, d \), then \( \pi_i(C) = S_i = \pi_i(B) \), so \( \langle \bar{r}, \bar{s} \rangle \in C_G^\pi_i(B)(\bar{c}, \bar{d}) = C_G^\pi_i(C)(\bar{c}, \bar{d}) \), where \( \bar{r}, \bar{s}, \bar{c}, \bar{d} \) are images in \( S_i \). If \( i \) does not split \( c, d \), then neither does \( i \) split \( r, s \), so
again $\langle r, s \rangle \in C g_{C}(c, d) = 0$. Then the Observation applies to show $\langle r, s \rangle \in C g_{C}(c, d)$. □

**Corollary** ([1]). If $A$ is a finite algebra of finite type for which $V(A)$ is congruence distributive, then $A$ is finitely based.

4. A group variety without DPSC

**Theorem 3.** The group variety $V(S_3)$ does not have DPSC.

**Proof:** We start from the observation that a variety $V$ with DPSC has “definable atomic congruences in finite members” in the sense that there is a congruence formula $\Gamma(u, v, x, y)$ for $V$ such that in any finite member $B$ of $V$, for each nontrivial congruence $C g_{B}(a, b)$ there is some atomic congruence $C g_{B}(r, s) \leq C g_{B}(a, b)$ for which $\Gamma(r, s, a, b)$ holds. Indeed, given $a, b$ we can choose $c, d$ as in the definition of DPSC and then an atomic congruence $C g_{B}(r, s)$ under $C g_{B}(c, d)$, so that $\Gamma(r, s, a, b)$ holds for $\Gamma(u, v, x, y) \equiv (\exists z, w)[\Gamma_1(z, w, x, y) \land \Gamma_2(u, v, z, w)]$, again a congruence formula.

If $V$ is a group variety, then principal congruences correspond to principal normal subgroups. For $a \in B \in V$, the elements of the principal normal subgroup $N(a)$ generated by $a$ are the products of conjugates of $a$ and $a^{-1}$. Let $V$ have finite exponent, so that mention of $a^{-1}$ can be omitted. By compactness, $V$ has definable atomic congruences in finite members when there is a bound $M$ such that for any finite member $B$ of $V$ and $a \in B$ with $a \neq 1$ there exists a minimal normal subgroup of $B$ generated by the product of at most $M$ conjugates of $a$.

We shall show that $V(S_3)$ lacks such a bound. Write $S_3 = \{1, c, c^2, b, bc, bc^2\}$, where $c^3 = 1, b^2 = 1$ and $cb = bc^2$. For future reference, observe that a conjugate $c^v = v^{-1}c v$ of $c$ for $v \in S_3$ depends only on the coset of $v$ modulo $A_3 = \{1, c, c^2\}$. For each $n$ let $B_n$ be the subgroup of $S_3^{2n}$ generated by $b_1^{(n)}, \ldots, b_n^{(n)}$, where $b_1^{(n)} = \langle 1, b, 1, b, \ldots \rangle$, $b_2^{(n)} = \langle 1, 1, b, b, 1, 1, b, b, \ldots \rangle$, and in general $b_k^{(n)}$ has alternating runs of $1$’s and $b$’s each of length $2^{k-1}$. Let $E_n$ be the larger subgroup of $S_3^{2n}$ generated by $b_1^{(n)}, \ldots, b_n^{(n)}$ and $c = \langle c, c, \ldots, c \rangle$.

First we show that the minimal normal subgroups of $E_n$ are all of the form $\{1\} \times \cdots \times \{1\} \times A_3 \times \{1\} \times \cdots \{1\}$. To establish principles let us examine $E_1$, which is generated by $\langle 1, b \rangle$ and $\langle c, c \rangle$. If $N$ is a nontrivial normal subgroup not of the stated form, then $N$ has an element $\langle x, y \rangle$ in which neither of $x, y$ is $1$. In the case where $y \in A_3$, we have $\{\langle x, y \rangle, \langle 1, b \rangle \} = \langle 1, y \rangle$ and $\{1\} < N(\langle 1, y \rangle) < N$. In the case where $y \notin A_3$, since $x \in A_3$ we have $\{\langle x, y \rangle, \langle c, c \rangle \} = \langle 1, c \rangle$ and $\{1\} < N(\langle 1, c \rangle) < N$. For $E_1$, these are the only cases, so that $N$
is not minimal. More generally, if \( N < E_n \) is a nontrivial normal subgroup not of the stated form, then \( N \) has some element \( x \) with two entries \( x_i, x_j \) neither of which is 1. In the case where both \( x_i, x_j \in A_3 \), as with \( E_1 \) we take the commutator of \( x \) with a generator \( b_k^{(n)} \) whose \( i \)-th and \( j \)-th entries differ. In the case where one of \( x_i, x_j \) is in \( A_3 \) and the other is not, we take the commutator with \( c \). In the case where \( x_i, x_j \notin A_3 \) (a case that does not occur for \( E_1 \)), the \( i \)-th and \( j \)-th entries of \([x, c] \in N \) are both \( c \), so we have arrived back at the first case. In all cases, we find that \( N \) is not minimal, showing that minimal normal subgroups of \( E_n \) do have the stated form.

Now suppose that there is a bound \( M \) as above for \( V \). Let \( n = M + 1 \) and consider any \( a \in N(c) < E_n \) other than the identity. We shall show that \( N(a) \) cannot be a minimal normal subgroup of \( E_n \). By assumption \( a \) is the product of at most \( M < n \) conjugates of \( c \), say \( a = c^{v_1} \cdots c^{v_k} \), where \( k < n \). Each conjugate \( c^{v(i)} \) is determined by the \( A_3 \)-cosets of the entries of \( v^{(i)} \); say \( v^{(i)}_j \in h^{(i)}_j A_3 \), where \( h^{(i)}_j \in \{1, b\} \). If we set \( h^{(i)} = \langle h^{(i)}_1, \ldots, h^{(i)}_{2n} \rangle \), we see \( h^{(i)} \in B_n \). A claim: The set of \( 2^n \) coordinate indices can be partitioned into nonsingleton blocks in such a way that the entries of each \( h^{(i)} \) are constant on each block. From this claim it follows that the entries of \( a \) are constant on each block. Then each entry value occurs in at least two coordinates and so \( a \) is not in any minimal normal subgroup as characterized above. We conclude that \( V(S_3) \) does not have definable atomic congruences in finite members.

To verify the claim, let \( H \) be the subgroup of \( B_n \) generated by \( h^{(1)}, \ldots, h^{(k)} \). Since \( B_n \) is an elementary 2-group with \( n \) independent generators and \( H \) has fewer than \( n \) generators, we have \( H < B_n \). The corresponding subgroup \( H' \) of the dual group \( \hat{B}_n \) is nontrivial and consists of characters that have value 1 on \( H \). Two characters are in the same coset of \( H' \) when they agree on \( H \). Now observe that from the construction of \( B_n \), the coordinate projections \( \pi_i : B_n \to \{1, b\} \) are the characters of \( B_n \) with \( \{1, b\} \) playing the role of \( \{-1, 1\} \). Thus the \( 2^n \) coordinate indices are partitioned into blocks of equal size (the cosets) such that each element of \( H \) has constant entries on each block. This is the partition to which the claim refers. \( \square \)

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References


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