Partially ordered sets

1. Definitions

A relation $\leq$ on a set $P$ is a *partial order relation* if

(a) $x \leq x$ (reflexivity)
(b) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry)
(c) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity)

$x \geq y$ means $y \leq x$; $x < y$ means $x \leq y$ and $x \neq y$; $x > y$ means $y < x$.

$\langle P, \leq \rangle$ is a *partially ordered set* (or *poset* or *partly ordered set* or *ordered set*) if $\leq$ is a partial order relation on $P$. (Generally we just say, “the partially ordered set $P$.”) In the following, $P$ and $Q$ refer to partially ordered sets.

The relation $\leq$ is a *total order relation* on $P$ if also

(d) for all $x, y$, either $x \leq y$ or $y \leq x$.

In this case, $\langle P, \leq \rangle$ is a *chain* or *totally ordered set* or *linearly ordered set*. (In contrast, if instead no two distinct elements are related, then $P$ is an *antichain*.)

In $P$, $a$ covers $b$ if $a > b$ and there is no $c$ with $a > c > b$.

The *Hasse diagram* of a finite partially ordered set $P$ is a diagram indicating the elements of $P$ by circles or dots, connected by lines that indicate the coverings in $P$. (No lines are drawn horizontal; a non-horizontal line from $b$ up to $a$ indicates that $a$ covers $b$.)

A map $f : P \rightarrow Q$ is said to be *isotone* if $f$ preserves order: $x \leq y \Rightarrow f(x) \leq f(y)$. It is possible, however, for an isotone map to take two unrelated elements to two related elements, or even to the same element.

A map $f : P \rightarrow Q$ is said to be an *isomorphism* if $f$ is one-to-one and onto and both $f$ and its inverse are isotone. In this case, $P$ and $Q$ are *isomorphic.*

*Note.* The best way to show that two partially ordered sets $P, Q$ are isomorphic is to define maps $f : P \rightarrow Q$ and $g : Q \rightarrow P$, show that $f$ and $g$ are isotone, and show that $f$ and $g$ are inverse to each other, in the sense that $g(f(p)) = p$ and $f(g(q)) = q$ for all $p \in P, q \in Q$. (It is *not* enough to define $f$ and show that $f$ is isotone, one-to-one, and onto.)

For partially ordered sets $P, Q$, the *direct product* partial order on the set $P \times Q$ is the coordinatewise ordering: $\langle p, q \rangle \leq \langle p', q' \rangle \iff p \leq p'$ and $q \leq q'$.
Figure 1: Some examples

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The Hasse diagram of $P \times Q$ can be drawn as a copy of $Q$ for each element of $P$, with $P$ used as a guide for the placement of the copies and for the coverings between them.

A direct product $P_1 \times \cdots \times P_n$ or $\prod_{i \in I} P_i$ is defined similarly.

2. Dilworth’s Theorem

2.1 Definition. The width of a partially ordered set $P$ is the cardinality of the largest antichain. (For example, a chain has width 1.)

Observation. If $P$ is the union of $n$ chains, then $P$ has width at most $n$.

2.2 Theorem (R. P. Dilworth) Let $P$ be a finite partially ordered set of width $n$. Then $P$ is a union of $n$ chains.

This is a kind of minimax theorem, in that it shows that the maximum size of an antichain in $P$ is the minimum number of chains whose union is $P$. There are a number of combinatorial consequences. Here is one:

Let $\rho$ be a binary relation between finite sets $A$ and $B$, i.e., $\rho \subseteq A \times B$. A matching of $A$ into $B$ is a one-to-one function $f : A \to B$ such that for all $a \in A$, $a \rho f(a)$.

2.3 Corollary (P. Hall’s matching theorem) Given $\rho$, a necessary and sufficient condition for the existence of a matching of $A$ into $B$ is that for each $k = 1, 2, \ldots$,

(*) any $k$ elements of $A$ are related to at least $k$ elements of $B$, in the sense that each of these elements of $B$ is related to at least one of the $k$ elements of $A$.

(In the proof, the disjoint union of $A$ and $B$ is made into a partially ordered set by declaring $a < b$ when $a \rho b$.)