Nonsingular matrices and transformations

This concerns square matrices and also transformations $T : V \rightarrow W$ where $V$ and $W$ have the same finite dimension.

Theorem. Let $A$ be an $n \times n$ matrix with entries in a field $F$, with the corresponding matrix transformation $\tau_A : F^n \rightarrow F^n$. Also let $T : V \rightarrow W$ be a linear transformation between vector spaces over $F$, both of dimension $n$, such that $T$ has matrix $A$ with respect to particular bases of $V$ and $W$. The following conditions are equivalent.

1. $\det A \neq 0$.
2. $A$ row-reduces to the $n \times n$ identity matrix.
3. $A$ has rank $n$ ("full rank", meaning the maximum rank possible).
4. The rows of $A$ are linearly independent.
5. The columns of $A$ are linearly independent.
6. Some system of linear equations with coefficient matrix $A$ has a unique solution.
7. Every system of linear equations with coefficient matrix $A$ has a unique solution.
8. $A$ has a right inverse, i.e., there is an $n \times n$ matrix $B$ with $AB = I$.
9. $A$ has a left inverse, i.e., there is an $n \times n$ matrix $B$ with $BA = I$.
10. $A$ has a two-sided inverse $A^{-1}$.
11. $A$ has nullity 0; in other words, nullspace $\tau_A = \{0\}$.
12. $Av = 0 \Rightarrow v = 0$
13. 0 is not an eigenvalue of $A$, i.e., there is no $v \neq 0$ with $Av = 0v$.
14. $\tau_A$ is one-to-one.
15. $\tau_A$ is onto.
16. $\tau_A$ is an isomorphism of $F^n$ with itself (an "automorphism" of $F^n$)
17. $\text{Nullspace}(T) = \{0\}$.
18. $T$ is one-to-one.

19. $T$ is onto.

20. $T$ is an isomorphism.

21. The matrix of $T$ with respect to any bases of $V$ and $W$ is nonsingular.

**Definition.** When any (and so all) of these conditions is satisfied, then $A$ is said to be *nonsingular*. Otherwise $A$ is *singular*.

(“Singular” means “special” or “unusual”, not to be confused with “single”. So a system of linear equations with *nonsingular* coefficient matrix has a *single* solution.)